Level I of the CFA® 2025 Exam

Study Notes - Quantitative Methods

Offered by AnalystPrep

Last Updated: Aug 1, 2024

Table of Contents

Learning Module 1: Rate and Return

LOS 1a: interpret interest rates as required rates of return, discount rates, or opportunity costs and explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk

The time value of money is a concept that states that cash received today is more valuable than cash received in the future. If a person agrees to receive payment in the future, he foregoes the option of earning interest if he invests that amount of money today.

An interest rate or yield, usually denoted by *r;* is a rate of return that reflects the connection between cash flows dated at different times.

Assume you currently possess $100. Next, consider depositing this money into a savings account, expecting it to grow to $110 after one year. Intuitively, the compensation required for deferring the consumption of $100 now in favor of receiving $110 in one year is $10 (equal to 110 minus 100). This compensation is equivalent to a 10% rate of return (calculated as 10 divided by 100).

There are three ways to interpret interest rates:

1. **Required rate of return**: The minimum return an investor expects to earn to accept an investment.
2. **Discount rate**: The rate used to discount future cash flows to allow for the time value of money (to determine the present value **equivalent** of some money to be received sometime in the future). Discount rates and interest rates are used almost interchangeably.
3. **Opportunity cost**: The value of the **best-forgone alternative**; the most valuable alternative investors give up when they choose what to do with money.

Determinants of Interest Rates

Economics postulates that the forces of supply and demand determine interest rates. In this case, the investors (lenders) supply the money, and the borrowers demand money for their consumption.

As such, interest is a reward a borrower pays for using an asset, usually capital, belonging to a lender. It is compensation for the loss or value depreciation occasioned by the use of the asset.

Therefore, an interest rate is composed of a real risk-free interest rate plus a set of four premiums that represent compensation for bearing distinct types of risk:

$$\text{Interest (r)} = \begin{array}{l} \text{Real risk-free interest rate} \\ + \text{ Inflation Premium} \\ + \text{ Default risk premium} \\ + \text{ Liquidity premium} \\ + \text{ Maturity premium} \end{array}$$

The Real Risk-free Interest Rate

The real risk-free interest rate is the single-period interest rate for a completely risk-free security if no inflation is expected. According to economic theory, the real risk-free rate reflects people's preferences for current compared to real future consumption.

Types of Risk Premiums

Inflation Risk Premium

Inflation risk is the loss of purchasing power of money as a result of the increase in prices of consumer goods.

The inflation premium compensates investors for expected inflation. It represents the average inflation rate expected over the maturity of the debt. The risk of a decrease in purchasing power validates the inflation risk premium.

Liquidity Risk Premium

Liquidity refers to the ease with which an investment can be converted into cash without significantly sacrificing market value.

The liquidity premium compensates investors for the risk of loss relative to an investment's fair value if the investment needs to be converted to cash quickly.

Default Risk Premium

Default risk describes a situation where a borrower may fail to repay borrowed funds as a result of bankruptcy. This might result in significant losses on the side of the lender.

The **default risk premium** compensates investors for the possibility that the borrower **will fail to make a promised payment** at the contracted time and in the contracted amount.

Maturity Risk Premium

The maturity risk premium is the additional return an investor requires for assuming interest rate and reinvestment risk resulting from a longer investment maturity timeline. Maturity risk premium increases with an increase in the maturity timeline. In other words, the longer the maturity timeline of an investment, the higher the maturity risk premium.

Nominal Risk-free Interest Rate

The nominal risk-free interest rate is defined as the sum of the real risk-free interest rate and the inflation premium. In other words, the nominal risk-free interest rate can be seen as the combination of the real risk-free rate plus an inflation premium, as shown by the following equation:

$$(1+\text{Nominal risk-free rate})=(1+\text{Real risk-free rate })(1+\text{Inflation premium})$$

The above equation is generally approximated as follows:

$$\text{Nominal risk-free rate}=\text{Real risk-free rate}+\text{Inflation premium}$$

Most rates quoted on short-term government debts can be taken as nominal risk-free interest rates over the respective maturity.

Question

Which of the following is *most likely* an interpretation of interest rate as a benefit foregone when investors spend money on current consumption instead of saving or investing?

A. Discount rate.

B. Opportunity cost.

C. Required rate of return.

Solution

The correct answer is **B**.

Opportunity cost is a key factor in interpreting interest rates. It refers to the interest foregone when investors opt for an alternate option, such as spending on current consumption instead of saving or investing.

A is incorrect. The discount rate is the interest rate used to discount future cash flows to reach the present value.

C is incorrect. The required rate of return is the minimum rate of return an investor would wish to earn to postpone current consumption.

LOS 1b: calculate and interpret different approaches to return measurement over time and describe their appropriate uses

Financial assets are primarily defined based on their return-risk characteristics. This helps when building a portfolio from all the assets available. Regarding returns, there are different ways of measuring returns.

Financial market assets generate two different streams of return: income through cash dividends or interest payments and capital gain or loss through financial asset price increases or decreases.

Some financial assets give only one stream of return. For instance, headline stock market indices typically report on price appreciation only. They do not include the dividend income unless the index specifies it is a “total return” series.

Holding Period Return

A holding period return is earned from holding an asset for a single specified period. The time period can be any specified period, such as a day, month, or ten years.

The general formula of the holding period return is given by:

$$R = \frac{(P_1 - P_0) + I_1}{P_0}$$

P_0 = Price of an asset at the beginning of the period (t=0).

P_1 = Price of an asset at the end of the period (t=1).

I_1 = Income received at the end of the period (t=1).

Example: Calculating Holding Period Return

An investor purchased 100 shares of a stock at $50 per share and held the investment for one year. During that period, the stock paid dividends of $2 per share. At the end of the year, the investor sold all the shares for $60 per share.

The holding period return is *closest to*:

Solution

In this case, we have:

$$\begin{aligned} P_0 &= 100 \text{ shares} \times \$50 \text{ per share} = \$5{,}000 \\ I_1 &= 100 \text{ shares} \times \$2 \text{ per share} = \$200 \\ P_1 &= 100 \text{ shares} \times \$60 \text{ per share} = \$6{,}000 \end{aligned}$$

Therefore,

$$R = \frac{(P_1 - P_0) + I_1}{P_0} = \frac{6{,}000 - 5{,}000 + 200}{5{,}000} = 24\%$$

Holding period returns can also be calculated for periods longer than a year. For instance, if we need to calculate the holding period return for a five-year period, we should compound the five annual returns as follows:

$$R = \frac{(P_5 - P_0) + I_{(1-5)}}{P_0}$$

Arithmetic Return

When we have assets for multiple holding periods, it is necessary to aggregate the returns into one overall return.

Denoted by $\bar{R}_i$ arithmetic mean for an asset i is a simple process of finding the average holding period returns. It is given by:

$$\bar{R}_i = \frac{R_{i,1} + R_{i,2} + \ldots + R_{i,T-1} + R_{iT}}{T} = \frac{1}{T} \sum_{t=1}^{T} R_{it}$$

Where:

R_{it} = Return of asset i in period t.

T = Total number of periods.

For example, if a share has returned 15%, 10%, 12%, and 3% over the last four years, then the arithmetic mean is computed as follows:

$$\bar{R}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it} = \frac{1}{4}(15\% + 10\% + 12\% + 3\%) = 10\%$$

Geometric Return

Computing a geometric mean follows a principle similar to the one used to compute compound interest. It involves compounding returns from the previous year to the initial investment's value at the start of the new period, allowing you to earn returns on your returns.

A geometric return provides a more accurate representation of the portfolio value growth than an arithmetic return.

Denoted by $\bar{R}_{Gi}$ the geometric return for asset i is given by:

$$\begin{aligned}\bar{R}_{Gi} &= \sqrt[T]{(1 + R_{i,1}) \times (1 + R_{i2}) \times \ldots \times (1 + R_{i,T-1}) \times (1 + R_{iT})} - 1 \\ &= \sqrt[T]{\prod_{t=1}^{T} (1 + R_t)} - 1\end{aligned}$$

Using the same annual returns of 15%, 10%, 12%, and 3% as shown above, we compute the geometric mean as follows:

$$\begin{aligned}\text{Geometric mean} &= [(1 + 15\%) \times (1 + 10\%) \times (1 + 12\%) \times (1 + 3\%)]^{\frac{1}{4}} - 1 \\ &= 9.9\%\end{aligned}$$

Note that the geometric return is slightly less than the arithmetic return. Arithmetic returns tend to be biased upwards unless the holding period returns are all equal.

Harmonic Mean

The harmonic mean is a measure of central tendency. It's especially useful for rates or ratios such as P/E ratios. Its formula is derived from the harmonic series, which is a specific mathematical sequence.

$$\bar{X}_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}},\ X_i > 0 \text{ for all } i = 1, 2, \ldots, n$$

The above formula is interpreted as the "harmonic mean of observations $X_1,\ X_2, \ldots,\ X_n$."

The harmonic mean is handy for averaging ratios when those ratios are consistently applied to a fixed quantity, resulting in varying unit numbers. For instance, it's applied in cost-averaging strategies where you invest a fixed amount of money at regular intervals.

Example: Calculating the Harmonic Mean

An investor is practicing cost averaging by investing in a particular stock over a period of three months. The investor decides to allocate different amounts of money each month. In the first month, the investor invests $2,000; in the second month, $3,000; and in the third month, $4,000. The share prices of the stock for these three months are $10, $12, and $15, respectively.

Calculate the average price paid per share for the three-month period.

Solution

Using the harmonic mean formula,

$$\bar{X}_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}} = \frac{3}{\frac{1}{10} + \frac{1}{12} + \frac{1}{15}} = 12$$

Trimmed and Winsorized Means

Trimmed and Winsorized means seek to lower the effect of outliers in a data set.

Trimmed Mean

The trimmed mean is a measure of central tendency in which we calculate the mean after excluding a small percentage of the lowest and highest values from the dataset.

For example, a data set consists of 10 observations: 12, 15, 18, 20, 22, 25, 27, 30, 35, and 40. We can calculate the trimmed mean after removing the highest and lowest values.

After removing these values, the remaining data set is: 15, 18, 20, 22, 25, 27, 30, and 35.

Now, let's calculate the trimmed mean by taking the average of these remaining values:

$$\frac{15+18+20+22+25+27+30+35}{8} = \frac{192}{8} = 24$$

Therefore, the trimmed mean of the given data set is 24.

Winsorized Mean

The Winsorized mean is a central tendency measure. It works by replacing extreme values at both ends of the data with the values of their closest observations. This process is similar to the trimmed mean. Essentially, it helps eliminate outliers in a dataset.

For example, consider a dataset of 12 observations: 8, 12, 15, 18, 20, 22, 25, 27, 30, 35, 40, and 50. We can calculate the Winsorized mean by replacing the lowest and highest values with those closest to the 10th and 90th percentiles, respectively. As such, the new values are **10**, 12, 15, 18, 20, 22, 25, 27, 30, 35, **37.5**, and 40, and the winsorized mean is:

$$\frac{10+12+15+18+20+22+25+27+30+35+37.5+40}{12} \approx 24.46$$

Question 2

What are the arithmetic mean and geometric mean, respectively, of an investment that returns 8%, -2%, and 6% each year for three years?

A. Arithmetic mean = 5.3%; Geometric mean = 5.2%.

B. Arithmetic mean = 4.0%; Geometric mean = 3.6%.

C. Arithmentic mean = 4.0%; Geometric mean = 3.9%.

Solution

The correct answer is **C**.

$$\text{Arithmetic mean} = \frac{8\% + (-2\%) + 6\%}{3} = 4\%$$

$$\text{Geometric mean} = [(1+8\%) \times (1+(-2\%)) \times (1+6\%)]^{1/3} - 1 = 3.9\%$$

LOS 1c: Compare the money-weighted and time-weighted rates of return and evaluate the performance of portfolios based on these measures

Money-weighted Rate of Return

The money-weighted return considers the money invested and gives the investor information on the actual investment return. Calculating money-weighted return is similar to calculating an investment's internal rate of return (IRR).

The money-weighted rate of return (MWRR) is like the portfolio's internal rate of return (IRR). It's the rate at which the present value of cash flows equals zero. In simple terms, it's a way to measure how well a portfolio is performing.

$$\sum_{t=0}^{T} \frac{CF_t}{(1+IRR)^t} = 0$$

Where:

T = Number of periods.

CF_t = Cash flow at time *t*.

IRR = Internal rate of return (or money-weighted rate of return).

The money-weighted rate of return (MWRR) looks at a fund's starting and ending values and all the cash flows in between. In an investment portfolio, cash inflows are a part of it. These inflows could be from deposits or investments made during a certain period. The MWRR considers these inflows and calculates the overall rate of return for the portfolio:

- The beginning value.
- Dividends/interest reinvested.
- Contributions made.

Cash outflows, on the other hand, refer to:

- Withdrawals made.
- Dividends or interest received.
- The final value of the fund.

Example 1: Calculating Money-weighted Rate of Return

An investor makes the following investments in a portfolio over a two-year period:

- At the beginning of year one, the investor invests $10,000.
- At the end of the first year, after the portfolio's value increases to $12,000, the investor adds $5,000, making the total portfolio value $17,000.
- At the end of the second year, the portfolio value further increases to $25,000.

The money-weighted rate of return for the investor’s portfolio is *closest* to:

Solution

We need to calculate the internal rate of return (IRR) considering the following cash flows:

- $\text{CF}_0 = -\$10,000$ (Initial investment)
- $\text{CF}_1 = -\$5,000$ (Additional investment at the end of year one)
- $\text{CF}_2 = +\$25,000$ (Final portfolio value at the end of year two)

To find the money-weighted rate of return, solve the equation for IRR:

$$\frac{\text{CF}_0}{(1+\text{IRR})^0}+\frac{\text{CF}_1}{(1+\text{IRR})^1}+\frac{\text{CF}_2}{(1+\text{IRR})^2}=\frac{-10,000}{1}+\frac{-5,000}{(1+\text{IRR})}+\frac{25,000}{(1+\text{IRR})^2}=0$$

Using BA II Plus Calculator, $\text{IRR} \approx 35.08\%$.

Example: Calculating Money-weighted Return for a Dividend-paying Stock

Calvin Hair purchased a share of Superior Car Rental Company for $85 at the beginning of the first year. He bought an additional unit for $87 at the end of the first year. At the end of the second year, he sold both shares at $90. During both years, Hair received a dividend of $4 per share, which was not reinvested.

Calculate the money-weighted return.

Solution

To calculate the money-weighted return in this example, we need to consider the timing and amounts of cash flows and their respective investment periods.

Step 1: Calculate the total investment at the beginning (t=0):

$$\text{Initial investment} = -\$85$$

Step 2: Calculate the total investment at $t = 1$:

$$\begin{aligned}\text{Initial investment + Additional investment} &= \$87 - \$4(\text{ Dividend received at the end of the first year, which is not reinvested)} \\ &= -\$83\end{aligned}$$

Step 3: Calculate the final portfolio value at $t = 2$:

$$\begin{aligned}\text{Number of shares sold} \times \text{Selling price} &= 2 \text{ shares} \times \$90 = \$180 + 8(\text{ Dividend received for both shares)} \\ &= \$188\end{aligned}$$

As such, we have:

$CF_0 = -85$.

$CF_1 = -83$.

$CF_2 = 188$.

Using the BA II Plus calculator, you will get $\text{IRR} = 7.71\%$, which is equivalent to the money-weighted rate of return.

Shortcomings of the Money-weighted Rate of Return

The money-weighted rate of return (MWRR) considers all cash flows, such as withdrawals or contributions. If an investment spans multiple periods, MWRR gives more importance to the fund's performance when the account is at its largest. This can be a problem for fund managers because it might make their performance seem worse due to factors they can't control.

Time-Weighted Rate of Return

The time-weighted rate of return (TWRR) calculates the compound growth of an investment. Unlike the money-weighted rate, it doesn't care about withdrawals or contributions. TWRR is like finding the average return of different time periods within your investment.

Steps of Calculating Time-weighted Rate of Return

Step 1: Value the portfolio immediately before any significant cash inflow or outflow of funds. Divide the evaluation period into subperiods based on dates of significant additions or withdrawals of funds.

Step 2: Compute the holding period return on the portfolio for each period.

Step 3: Compound or link the holding period returns to the annual rate of return, which is the time-weighted rate of return.

$$\text{TWRR} = (1 + \text{HPR}_1 \times (1 + \text{HPR}_2) \times (1 + \text{HPR}_3) \ldots \times (1 + \text{HPR}_{n-1}) \times (1 + \text{HPR}_n) - 1$$

If the evaluation period is more than one year, compute the geometric mean of the annual returns to get the time-weighted return for the investment period.

$$\bar{R}_{Gi} = \sqrt[n]{(1 + \text{HPR}_1) \times (1 + \text{HPR}_1) \ldots \times (1 + \text{HPR}_n)} - 1$$
$$= [(1 + \text{HPR}_1) \times (1 + \text{HPR}_1) \ldots \times (1 + \text{HPR}_n)]^{\frac{1}{n}} - 1$$

Example: Calculating the Time-Weighted Rate of Return (Period More than one year)

An investor purchases a share of stock at t = 0 for $200. At the end of the year (at t = 1), the investor purchases an additional share of the same stock, this time for $220. She then sells both shares at the end of the second year for $230 each. She also received annual dividends of $3 per share at the end of each year. Calculate the annual time-weighted rate of return on her investment.

Solution

First, we break down the two years into two one-year periods.

Holding period 1:

Beginning value = 200.

Dividends paid = 3.

Ending value = 220.

Holding period 2:

Beginning value = 440 (2 shares × 220)

Dividends paid = 6 (2 shares × 3)

Ending value = 460 (2 shares × 230)

Secondly, we calculate the HPR for each period:

$$HPR_1 = \frac{(220 - 200 + 3)}{200} = 11.5\%$$
$$HPR_2 = \frac{(460 - 440 + 6)}{440} = 5.9\%$$

Lastly, we need to find the geometric mean of the HPRs since we are dealing with a period of more than a year.

$$\begin{aligned} TWRR &= [(1 + HPR_1) \times (1 + HPR_1) \ldots \times (1 + HPR_n)]^{\frac{1}{n}} - 1 \\ &= (1.115 \times 1.059)^{0.5} - 1 = 8.7\% \end{aligned}$$

Example: Calculating the Time-weighted Rate of Return (Period Less

than One Year)

The beginning value of a portfolio as of January 1, 2020, was $1,000,000. On February 10, the portfolio's value was $1,100,000, including an additional contribution of the $50,000 injected into the portfolio on this date. The portfolio's ending value at the beginning of April was $1,350,000.

The time-weighted rate of return is *closest to:*

Solution

The time-weighted return is calculated as follows:

$$\begin{aligned} HPR_1 &= \frac{V_1 - V_0}{V_0} = \frac{(1,100,000 - 50,000) - 1,000,000}{1,000,000} = 5\% \\ HPR_2 &= \frac{V_2 - V_1}{V_1} = \frac{1,350,000 - 1,100,000}{1,100,000} = 22.73\% \\ \Rightarrow TWRR &= (1 + HPR_1) \times (1 + HPR_1) - 1 \\ &= 1.05 \times 1.2273 - 1 = 28.87\% \end{aligned}$$

Question

A chartered analyst buys a share of stock at time t = 0 for $50. At t = 1, he purchases an extra share of the same stock for $53. The share gives a dividend of $0.50 per share for the first year and $0.60 per share for the second year. He sells the shares at the end of the second year for $55 per share. Calculate the annual time-weighted rate of return.

A. 5.90%.

B.12.24%.

C. 7.00%.

The correct answer is **A.**

We have two one-year holding periods:

$$\begin{array}{cc} HP_1 & HP_2 \\ P_0 = 50 & P_0 = 106 \\ D = 0.5 & D = 1.2 \\ P_1 = 53 & P_1 = 110 \end{array}$$

We now calculate the holding period returns:

$$HPR_1 = \frac{(53 - 50 + 0.5)}{50} = 7\%$$
$$HPR_2 = \frac{(110 - 106 + 1.2)}{106} = 4.9\%$$
$$\Rightarrow TWRR = 1.07 \times 1.049 - 1 = 12.24\%$$

Therefore,

$$\text{Annual TWRR} = (1 + 0.1224)^{0.5} - 1 = 5.9\%$$

LOS 1d: Calculate and interpret annualized return measures and continuously compounded returns and describe their appropriate uses

To compare returns over different timeframes, we need to annualize them. This means converting daily, weekly, monthly, or quarterly returns into annual figures.

Non-Annual Compounding

Interest may be paid semiannually, quarterly, monthly, or even daily – interest payments can be made more than once a year. Consequently, the present value formula can be expressed as follows when there are multiple compounding periods in a year:

$$PV = FV_N \left(1 + \frac{R_S}{m}\right)^{-mN}$$

Where:

m = Number of compounding periods in a year.

R_s = Quoted annual interest rate.

N = Number of years.

Example: Calculating the Present Value of a Lump Sum (More than One Compounding Period)

Jane Doe wants to invest money today and have it become $500,000 in five years. The annual interest rate is 8%, and it's compounded quarterly. How much should Jane invest right now?

Using the formula above:

$FV_N = \$500,000.$

$R_S = 8\%.$

$m = 4.$

$R_S/m = \frac{8\%}{4} = 2\% = 0.02.$

$N = 5.$

$mN = 4 \times 5 = 20.$

Therefore,

$$PV = FV_N \left(1 + \frac{R_S}{m}\right)^{-mN} = \$500,000 \times (1.02)^{-20} = \$336,485.67$$

<u>Using BA II *Plus* Calculator</u>:

- Press the [2nd] button, then the [FV] button to clear the financial registers. The display should show "CLR TVM."
- Enter the future value (FV). This is the amount Jane wants to have in five years, which is $500,000. To do this, type "500000" and press the [FV] button.
- Enter the interest rate (I/Y). This is the annual interest rate, which is 8%. However, since interest is compounded quarterly, we need to divide this by 4. To do this, type "8", press the [÷] button, type "4", then press the [ENTER] button, and finally press the [I/Y] button.
- Enter the number of periods (N). This is the number of quarters in five years, which is 5*4 = 20. To do this, type "20" and press the [N] button.
- Compute the present value (PV). To do this, press the [CPT] and then the [PV] buttons. The display should show the amount Jane needs to invest today, approximately $336,485.49.

Annualized Returns

To annualize a return for a period shorter than a year, you need to account for how many times that period fits into a year. For example, if you have a weekly return, you would compound it 52 times because there are 52 weeks in a year.

Generally, we can annualize the returns using the following formula:

$$\text{Return}_{\text{annual}} = (1 + \text{Return}_{\text{period}})^{c} - 1$$

Where:

$\text{Return}_{\text{period}}$ = Quoted return for the period.

c = Number of periods in a year.

Example: Annualizing Returns

If the monthly return is 0.7%, then the compound annual return is:

$$\begin{aligned}\text{Return}_{\text{annual}} &= (1 + \text{Return}_{\text{monthly}})^{12} - 1 \\ &= (1.007)^{12} - 1 = 0.0873 = 8.73\%\end{aligned}$$

For a period of more than one year, for example, a 15-month return of 16% can be annualized as:

$$\begin{aligned}\text{Return}_{\text{annual}} &= (1 + \text{Return}_{15\text{ month}})^{\frac{12}{15}} - 1 \\ &= (1.16)^{\frac{4}{5}} - 1 = 12.61\%\end{aligned}$$

We may apply the same procedure to convert weekly returns to annual returns for comparison with weekly returns.

$$\text{Return}_{\text{annual}} = (1 + \text{Return}_{\text{weekly}})^{52} - 1$$

For comparison with weekly returns, we can convert annual returns to weekly returns by making $(\text{Return}_{\text{weekly}})^{52}$ the subject of the formula.

Example: Comparing Investments by Annualizing Returns

An investor is evaluating the returns of two recently formed bonds. Selected return information on the bonds is presented below:

Bond	Time Since Issuance	Return Since Issuance (%)
A	120 days	2.50
B	8 months	6.00

Annualized Return Calculation

To compare the annualized rate of return for both bonds, you can use the formula for annualizing returns based on different time periods:

Annualized Return = $(1+\frac{\text{Return Since Issuance}}{100})^{\frac{365}{\text{Time Since Issuance}}}-1$

Let's calculate the annualized returns for both bonds:

For Bond A:

Time Since Issuance = 120 days

Return Since Issuance = 2.50%

Annualized Return for Bond A = $(1+\frac{2.50}{100})^{\frac{365}{120}}-1$

Annualized Return for Bond A = $(1+0.025)^{3.0417}-1$

Annualized Return for Bond A = 1.079847 - 1 = 0.079847 or 7.98%

For Bond B:

Time Since Issuance = 8 months = 240 days

Return Since Issuance = 6.00%

Annualized Return for Bond B = $(1+\frac{6.00}{100})^{\frac{365}{240}}-1$

Annualized Return for Bond B = $(1+0.06)^{1.5208}-1$

Annualized Return for Bond B = 1.092751 - 1 = 0.092751 or 9.28%

Comparing the annualized returns:

Bond A has an annualized return of approximately 7.98%.

Bond B has an annualized return of approximately 9.28%.

Therefore, Bond B has a higher annualized rate of return compared to Bond A.

Continuously Compounded Returns

The continuously compounded return is calculated by taking the natural logarithm of one plus the holding period return. For example, if the monthly return is 1.2%, you'd calculate it as ln(1.012), which equals approximately 0.01192.

Generally, continuously compounded from t to $t+1$ is given by:

$$r_{t,t+1} = \ln\left(\frac{P_{t+1}}{P_t}\right) = \ln\left(1 + R_{t,t+1}\right)$$

Assume now that the investment horizon is from time $t = 0$ to time $t = T$ then the continuously compounded return is given by:

$$r_{0,T} = \ln\left(\frac{P_T}{P_0}\right)$$

If we apply the exponential function on both sides of the equation, we have the following:

$$P_T = P_0 e^{r_{0,T}}$$

Note that $\frac{P_T}{P_0}$ can be written as:

$$\frac{P_T}{P_0} = \left(\frac{P_T}{P_{T-1}}\right)\left(\frac{P_{T-1}}{P_{T-2}}\right)\ldots\left(\frac{P_1}{P_0}\right)$$

If we take natural logarithm on both sides of the above equation:

$$\ln\left(\frac{P_T}{P_0}\right) = \ln\left(\frac{P_T}{P_{T-1}}\right) + \ln\left(\frac{P_{T-1}}{P_{T-2}}\right) + \ldots + \ln\left(\frac{P_1}{P_0}\right)$$
$$\Rightarrow r_{0,T} = r_{T-1,T} + r_{T-2,T-1} + \ldots + r_{0,1}$$

Therefore, the continuously compounded return to time T is equivalent to the sum of one-period continuously compounded returns.

Question

The weekly return of an investment that produces an annual compounded return of 23% is *closest to*:

A. 0.40%.

B. 0.92%.

C. 0.41%.

The correct answer is A.

Recall that:

$$\text{Return}_{\text{annual}} = (1 + \text{Return}_{\text{weekly}})^{52} - 1$$

We can rewrite the above equation as follows:

$$\begin{aligned}\text{Return}_{\text{weekly}} &= (1 + \text{Return}_{\text{annual}})^{\frac{1}{52}} - 1\\ &= (1 + 0.23)^{\frac{1}{52}} - 1\\ &\approx 0.40\%\end{aligned}$$

LOS 1e: calculate and interpret major return measures and describe their appropriate uses

Other Return Measures

Gross and Net Return

The gross return is what an asset manager earns before subtracting various costs such as management fees, custody fees, taxes, and other administrative expenses. However, it does account for trading costs such as commissions.

Gross return does not consider management or administrative costs. For this reason, it is a suitable metric for assessing and comparing the investment expertise of asset managers.

Net return is a metric for how much an investment has earned for the investor. It considers all administrative and management costs that reduce an investor's return.

Pre-tax and After-tax Nominal Return

Unless otherwise stated, all returns are nominal pre-tax returns in general. Depending on the jurisdiction, different rates apply to capital gains and income. Long-term and short-term taxes may also be applied to capital gains.

The after-tax nominal return is determined by subtracting any tax deductions applied to dividends, interest, and realized gains from the total return.

Real Returns

Returns are typically presented in nominal terms, which consist of three components: the real risk-free return as compensation for postponing consumption, inflation as compensation for the loss of purchasing power, and a risk premium. Real returns are useful in comparing returns over different periods, given that inflation rates vary over time.

Recall the relationship between the nominal rate and the real rate:

$$(1+\text{ Nominal Risk-free rate})=(1+\text{Real risk free rate})(1+\text{Inflation premium})$$

We can find the connection between nominal and real returns by considering the real risk-free rate of return and the inflation premium. This relationship can be expressed as:

$$(1+\text{Real Return})=\frac{(1+\text{Real risk-free rate})(1+\text{Risk premium})}{1+\text{Inflation premium}}$$

Real returns become particularly useful when you want to compare returns across various time periods and different countries. This is especially important when returns are shown in local currencies and when inflation rates vary from one country to another.

After-tax real return is the amount the investor receives as payment for delaying consumption and taking on risk after paying taxes on investment.

Leveraged Returns

If an investor uses derivative instruments within a portfolio or borrows money to invest, then leverage is introduced into the portfolio. The leverage amplifies the returns on the investor's capital, both upwards and downwards.

The leveraged return considers the actual return on the investment and the cost of the borrowed money. The cost of borrowing and financing fees are subtracted from the overall return produced by the investment to determine the leveraged return.

Using the borrowed capital (debt) increases the size of the leveraged position by the additional borrowed capital.

Intuitively, the leveraged return is given by:

$$\begin{aligned}R_L&=\frac{\text{Portfolio return}}{\text{Portfolio equity}}\\&=\frac{[R_P\times(V_E+V_B)-(V_B\times r_D)]}{V_E}\\&=R_P+\frac{V_B}{V_E}(R_P-r_D)\end{aligned}$$

Where:

R_L = Return earned on the leveraged portfolio.

R_P = Total investment return earned on the leveraged portfolio.

V_B = Value of debt in the portfolio.

V_E = Value of equity in the portfolio.

r_D = Borrowing cost on debt.

Example: Calculating Leveraged Return

For a $250,000 equity portfolio with an annual 9% total investment return, 40% financed by debt at 6%, the leveraged return would be:

$$R_L = R_P + \frac{V_B}{V_E}(R_P - r_D) = 9\% + \frac{\$100,000}{\$150,000}(9\% - 6\%) = 11\%$$

Question

A $7,500,000 equity portfolio is 35% financed by debt at a cost of 5% per annum. If the equity portfolio generates a 9% annual total investment return, the leverage return is *closest* to:

A. 11.15%.

B. 14.00%.

C. 8. 25%.

The correct answer is **A**.

$$\begin{aligned} R_L &= R_P + \frac{V_B}{V_E}(R_P - r_D) \\ &= 9\% + \frac{\$2,625,000}{\$4,875,000}(9\% - 5\%) = 11.15\% \end{aligned}$$

Learning Module 2: The Time Value of Money in Finance

LOS 2a: calculate and interpret the present value (PV) of fixed-income and equity instruments based on expected future cash flows

The time value of money (TVM) is a fundamental financial concept. It emphasizes that a sum of money is worth more in the present than in the future. There are three key reasons supporting this principle:

- **The concept of opportunity** cost suggests that money available today can be invested and generate interest, increasing its value over time. By delaying the use of money, one forgoes potential investment opportunities and the growth they offer.
- **Inflation** poses a threat to the purchasing power of money in the future. Due to inflation, the same amount of money may buy fewer goods or services in the future compared to its present value. Consequently, having money now is advantageous since its purchasing power diminishes as time progresses.
- There is an element of **uncertainty regarding future cash flows**. Unexpected events or circumstances may prevent the receipt of money as planned, rendering it less reliable. Until the money is obtained, there is a level of uncertainty attached to its availability and utility.

Time value of money calculations allow us to establish the future value of a given amount of money.

Key Components of Time Value of Money

- **Discount rate or interest rate**: The rate of discounting or compounding that you apply to an amount of money to calculate its present or future value.
- **Time periods**: The whole number of time periods over which the present or future value of a sum is being calculated. These periods can be annually, semi-annually, quarterly, monthly, weekly, etc.

- **Present value (PV)**: The amount of money you have today (or at time T = 0) is referred to as the present value.
- **Future value (FV):** The accumulated amount of money you get after investing the original sum at a specific interest rate and for a given time period, say, two years.

Fundamental Formulas in Time Value of Money Calculations:

Let,

FV = Future value.

PV = Present value.

r = Stated discount rate per period.

N = Number of periods (Years).

Then the future value (FV) of an investment is given by:

$$FV = PV(1 + r)^N$$

If N is large such that $N \to \infty$ the initial cashflow is compounded continuously:

$$FV = PVe^{rN}$$

To find the present value of the investment, we rewrite the above formula so that:

$$PV = FV(1 + r)^{-N}$$

And for the continuous compounding, we have,

$$PV = FV_t e^{-rN}$$

Example: Calculating the Present Value of continuously Compounded Cashflows

A fund continuously accumulates to $4,000 over ten years at a 10% annual interest rate.

Calculate the closest present value of this fund.

Solution

From the question, FV=4,000, r_s=10%, N=10

So,

$$PV = FVe^{-Nr_s} = \$4,000 \times e^{-10 \times 0.1} = \$1,471.5178$$

Frequency of Compounding

When the frequency of compounding is more than once per year (quarterly, monthly, etc.), the formulas are analogously defined as:

$$FV_N = PV(1 + \frac{r_s}{m})^{mN}$$

Where:

m = Number of compounding periods per year.

N = Number of years.

r_s = Annual stated rate of interest.

Intuitively, the formula for the PV is given by:

$$PV = FV(1 + \frac{r_s}{m})^{-mN}$$

In the following discussion, we shall let $t = mN$ denote the number of compounding periods and $\frac{r_s}{m} = r$ denote the stated discount rate per period.

Calculation using a Financial Calculator

For calculating FV and PV using the BA II Plus™ Financial Calculator, use the following keys:

N = Number of compounding periods.

I/Y = Rate per period.

PV = Present value.

FV = Future value.

PMT = Payment.

CPT = Compute.

It is important to note that the sign of PV and FV will be opposite. For example, if PV is negative, then FV will be positive. Generally, an inflow is entered with a positive sign, while an outflow is entered as a negative sign in the calculator.

Time Value of Money in Fixed-income Instruments

Fixed-income instruments are debt securities where an issuer borrows money from an investor (lender) in exchange for a promised future payment. Examples of fixed-income instruments are bonds, loans, and notes.

The market discount rate for fixed-income instruments is also known as yield-to-maturity (YTM). It's the interest rate investors require to invest in a specific fixed-income instrument.

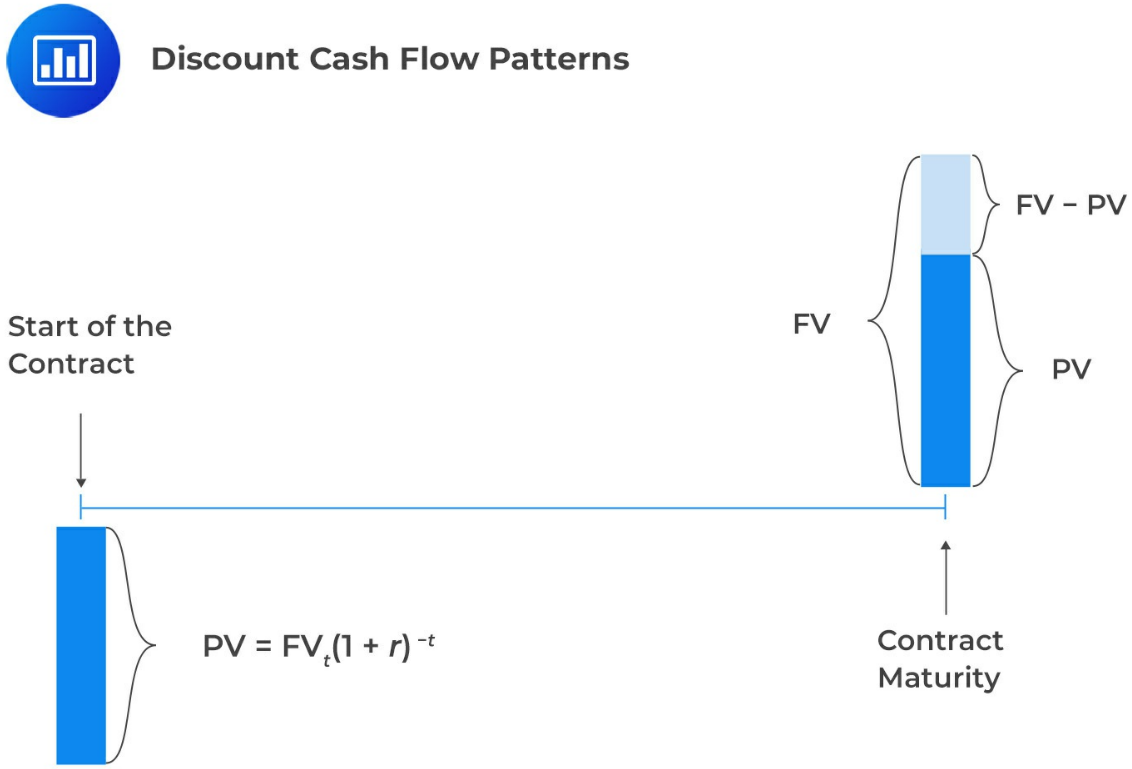
Cash Flow Patterns Associated with Fixed-Income Instruments

The cash flows in fixed-income instruments occur in three general patterns: Discount, periodic interest, and level payments.

Discount Cash Flow Patterns

For discount cashflow patterns, an investor pays an initial discounted price (PV) for the instrument (such as a bond or a loan) and gets one payment (FV) at the end maturity. The investor's return is the interest earned, that is, the difference between the initial price and principle ($FV - PV$).

The discount bonds are also called zero-coupon bonds – they do not have periodic interest payments.



The price of a discount bond can be calculated using the formula for the present value (PV) of a single cash flow, which is as follows:

$$PV = FV_t(1+r)^{-t}$$

Where:

FV = Future value.

PV = Present value.

r = Stated discount rate per period.

t = Number of compounding periods.

Example: Calculating the Future Value of a Zero-Coupon Bond

Assume Chad invests $8,000 in a zero-coupon bond that yields 8% annually and matures in four years. The maturity value of this bond is *closest to*:

Solution

Recall that:

$$FV = PV(1 + r)^t$$

In this case, we have PV=8,000, r=8%, t=4 so that:

$$FV = 8,000(1 + 8\%)^4 = 10,883.91$$

Using the BA II Plus[TM] Financial Calculator

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
2[N]	Years/periods	$N = 4$
10[1/Y]	Set interest rate	$PV = -8,000$
0[PMT]	Set payment	$PMT = 0$
[CPT][FV]	Compute future value	$FV = 10,883.91$

Note that zero-coupon bonds can be issued at negative interest rates. In this case, the price (PV) of the bond is higher than the face value (FV).

Example: Calculating the Price of a Discount Bond Issued at Negative Interest Rates

In January 2018, the Swiss government issued 15-year sovereign bonds at a negative yield of -0.08%. The present value (PV) of the bond per CHF100 of principal (FV) at the time of issuance is *closet to*:

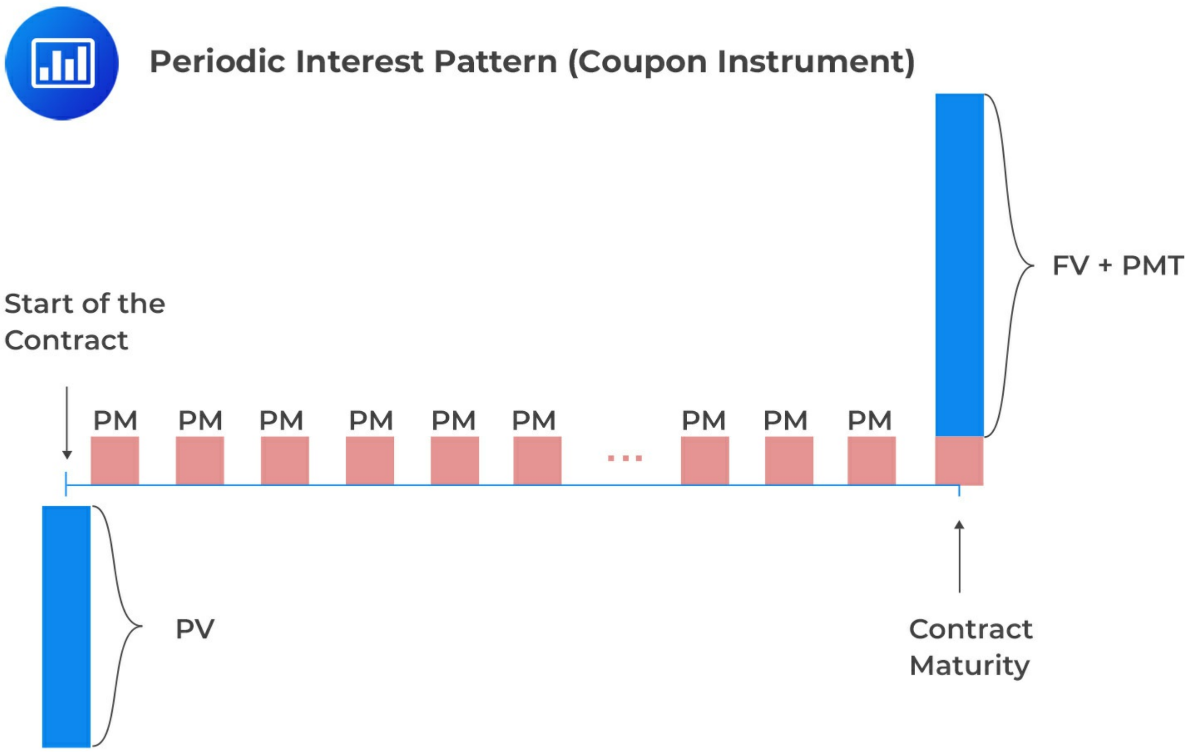
Solution

Recall for a zero-coupon bond,

$$\begin{aligned} PV &= FV_t(1 + r)^{-t} \\ &= 100(1 - 0.0008)^{-15} = 101.21 \end{aligned}$$

Periodic Interest Pattern (Coupon Instrument)

A coupon instrument is a fixed-income investment. It includes periodic cash flows called coupons and repays the principal at maturity. People often use these in coupon bond investments. These instruments have a set schedule with regular, equal payments.



The pricing of a coupon bond involves calculating its present value (PV) based on the market discount rate. The general formula for calculating the bond's price is derived from the discounted cash flow model. It considers the coupon payments (PMTs) and the final principal payment (FV) at maturity. The bond's price is determined by discounting each cash flow using the market discount rate (r).

The formula used to calculate the present value (PV) of a coupon bond is as follows:

$$\text{PV(Coupon Bond)} = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \ldots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

Where:

PMT = Coupon payment.

FV = Future value.

r = Market discount rate (YTM).

N = Number of periods.

Example 1: Pricing a Coupon Bond on an Annual Basis

Suppose we have a 5-year bond with a face value of $1,000 and an annual coupon rate of 5%. The market discount rate is 6%. The bond's price is *closest to*:

Solution

$$\text{PV(Coupon Bond)} = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \cdots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

In this case, we have PMT=5% of $1,000=$50, r=6%, N=5 years, FV=$1,000 so that:

$$\begin{aligned} PV &= \frac{\$50}{(1+0.06)^1} + \frac{\$50}{(1+0.06)^2} + \frac{\$50}{(1+0.06)^3} + \frac{\$50}{(1+0.06)^4} \\ &+ \frac{(\$50 + \$1,000)}{(1+0.06)^5} \\ PV &= \$47.17 + \$44.50 + \$41.98 + \$39.60 + \$784.62 = \$957.88 \end{aligned}$$

Therefore, the price of the bond would be $957.88

You could use a BA II Plus Calculator to solve the above question:

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
5[N]	Years/periods	N = 5
6[1/Y]	Set interest rate	I/Y = 6.00
50[PMT]	Set payment	PMT = 50.00
1000[FV]	Set the face value	FV = 1000.00
[CPT][PV]	Compute the present value	PV = −957.88

Example 2: Pricing a Coupon Bond With a Single Cash Flow on a semi-annual Basis

Suppose an investor has a 2-year bond with a face value of $1000 and an annual coupon rate of 6%, paid semi-annually. The market discount rate is 5%. The price of the bond is *closest to*:

Solution

Recall that:

$$\text{PV(Coupon Bond)} = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \ldots + \frac{(PMT_N + FV_N)}{(1+r)^N}$$

Where:

PMT = Coupon payments ($\$1,000 \times \frac{6\%}{2}$) = $30 *in this case.*

FV = Future value ($1,000 *in this case*).

r = Market discount rate (YTM), ($\frac{(5\%)}{2}$ = 2.5%) *per period in this case.*

N = Number of periods (4 *periods in this case*).

Plugging these values into the formula, we get:

$$PV = \frac{\$30}{(1.025)^1} + \frac{\$30}{(1.025)^2} + \frac{\$30}{(1.025)^3} + \frac{(\$30 + \$1,000)}{(1.025)^4}$$
$$PV = \$29.27 + \$28.55 + 27.85 + \$933.13 = \$1,018.81$$

Therefore, the bond's price is $1,018.81

You can easily use the BA II Plus calculator (or any other allowed financial calculator) to solve the above question.

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
4[N]	Years/periods	N = 4
2.5[1/Y]	Set interest rate	I/Y = 2.50
30[PMT]	Set payment	PMT = 30.00
1000[FV]	Set the face value	FV = 1000.00
[CPT][PV]	Compute the present value	PV = −1,018.81

Perpetual Bonds

Perpetual bonds are rare types of coupon bonds that do not have a stated date of maturity. They are generally issued by firms seeking equity-like financing and usually include redemption provisions.

The formula present value of perpetual bonds is obtained as follows: As $N \rightarrow \infty$, the formula for calculating PV of coupon changes as follows:

$$\begin{aligned} &\text{PV (perpetual bond)} \\ &= \lim_{(N \rightarrow \infty)} \left[\frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \cdots + \frac{(PMT_N + FV_N)}{(1+r)^N}\right] \\ &= \frac{PMT}{r} \end{aligned}$$

So, the present value of a perpetuity is given by:

$$PV = \frac{PMT}{r}$$

Example: Perpetual Bond

In 2021, XYZ Financial (the holding company for XYZ Bank) issued \$500 million in perpetual bonds with a 4.00 percent semi-annual coupon. Calculate the bond's yield to maturity (YTM) if the market price was \$98.50 (per \$100).

Solution

Recall,

$$PV = \frac{PMT}{r}$$

Hence,

$$r = \frac{PMT}{PV}$$

To solve this problem, we first need to calculate the semi-annual coupon payment, which is,

$$\text{PMT(semi-annual coupon payment)} = \frac{\$100 \times 4\%}{2} = \$2, \text{PV} = \$98.50$$

Therefore,

$$r = \frac{\$2}{\$98.50} = 0.0203 = 2.03\%$$

The annualized yield-to-maturity is:

$$r = 0.0203 \times 2 \approx 4.06\%$$

Level Payments (Annuity Instruments) Patterns

An annuity is a finite series of cash flows, all with the same value. **A fixed-income instrument** with annuity payments provides a stream of periodic equal cash inflows over a finite period.

The level payments consist of interest and principal payments. Fixed income instruments with level payments include fully amortizing loans such as mortgages.

There are two types of annuities: ordinary annuities and annuities due. Annuity due is a type of annuity where payments start immediately at the beginning of time, at time $t = 0$. In other words, payments are made at the beginning of each period.

On the other hand, an ordinary annuity is an annuity where the cashflows occur at the end of each period. Such payments are said to be made in arrears (beginning at time t = 1). We shall consider ordinary annuity in this section.

Ordinary Annuity

Remember that in an ordinary annuity, the series of payments does not begin immediately. Instead, payments are made at the end of each period. It is further worth noting that the present value of an annuity is equal to the sum of the current value of each annuity payment:

$$
\begin{aligned}
PV &= A(1+r)^{-1} + A(1+r)^{-2} + \ldots + A(1+r)^{-N-1} + A(1+r)^{-N} \\
&= A(1+r)^{-1} + (1+r)^{-2} + \ldots + (1+r)^{-(N-1)} + (1+r)^{-N} \\
PV &= A\frac{1-(1+r)^{N}}{r}
\end{aligned}
$$

Where:

A = Periodic cash flow.

r = Market interest rate per.

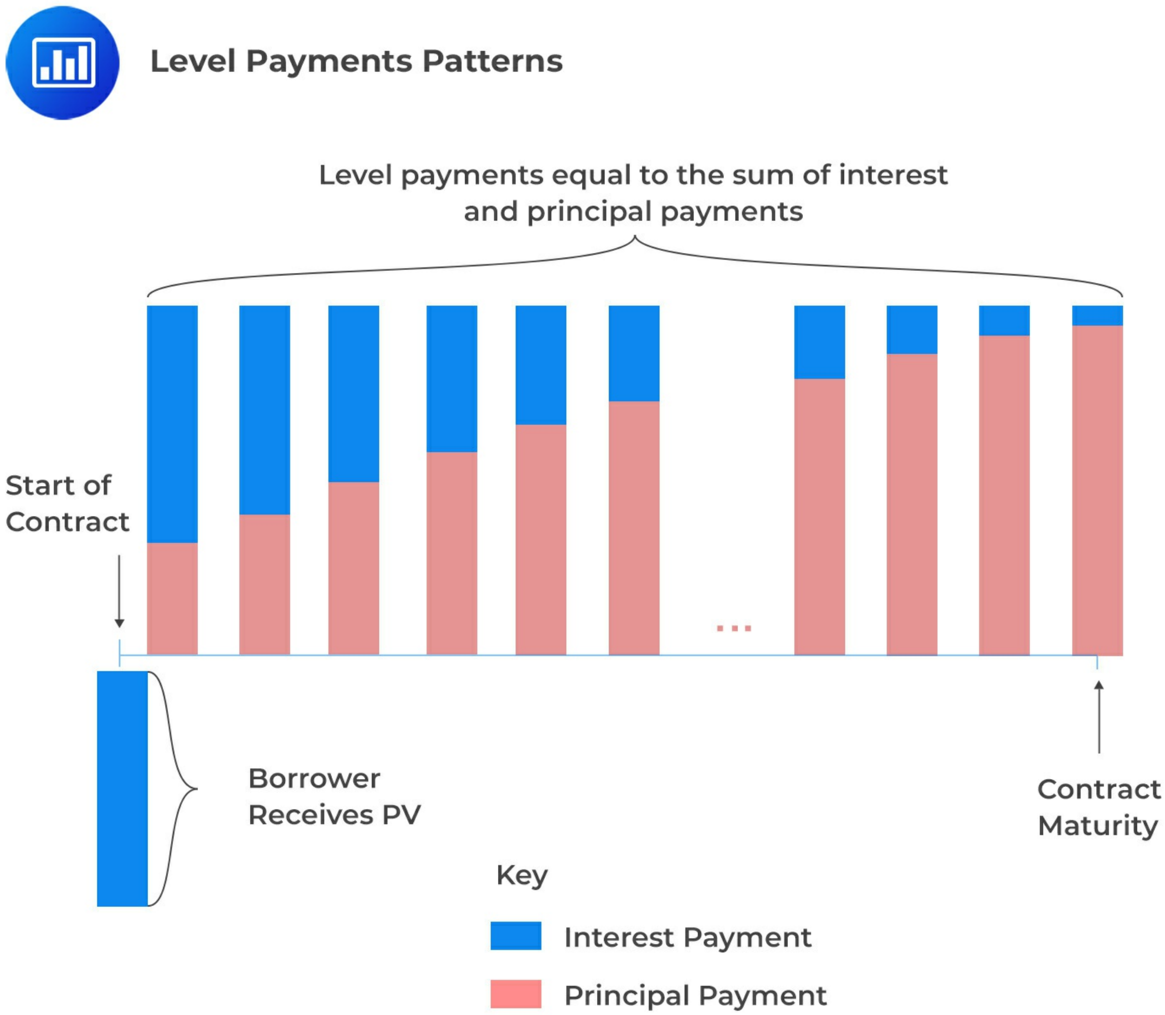
PV = Present value/ Principal Amount of the loan or bond.

N = Number of payment periods.

The periodic annuity is calculated as follows:

$$A = \frac{r(PV)}{(1-(1+r)^{-N}}$$

Consider a fully amortizing mortgage loan. In this case, the borrower receives the mortgage loan now and promises to make periodic payments equal to the sum of interest and principal payments.



Note that the periodic mortgage payment is constant, but the proportion of the interest payment decreases while the principal payment increases.

The cash flow pattern of a fully amortizing mortgage follows the pattern of an ordinary annuity with a series of equal cash flows. As such, the periodic annuity (periodic payment) of a fully amortizing mortgage is given by:

$$A = \frac{r(PV)}{1-(1+r)^{-t}}$$

Where:

A = Periodic cash flow.

r = Market interest rate per period.

PV = Present value or principal amount of loan or bond.

t = Number of payment periods.

Example: Calculating the Periodic Payment of a Mortgage

Jake is looking to secure a fixed-rate 25-year mortgage to finance 75% of the value of an $800,000 residential property. If the annual interest rate on the mortgage is 4.5%, Jake's monthly mortgage payment is *close to*:

Solution

Remember,

$$A = \frac{r(PV)}{1-(1+r)^{-t}}$$

Where:

A = Periodic cash flow.

r = Market interest rate per period.

PV = Present value/ Principal Amount of the loan or bond.

t = Number of payment periods.

In this case, we have:

- $r = 0.375\% \left(= \frac{4.5\%}{12}\right)$
- $N = 300 \text{ months}(= 25 \text{ years} \times 12 \text{ months/year})$
- $PV = \$600,000(= 75\% \times \$800,000)$

Plugging these values into the formula, we get:

$$A = \frac{0.00375 \times \$600,000}{1-(1+0.00375)^{-300}} = \$3,334.995 \approx \$3,335$$

Using a BA II Plus financial calculator:

Steps	Explanation	Display
[2nd][QUIT]	Return to standard calc Mode	0
[2nd][CLR TVM]	Clears TVM Worksheet	0
300[N]	Years/periods	N = 300
0.375[1/Y]	Set interest rate	I/Y = 0.375
−600,000[PV]	Set the present value of the mortgage	PV = −600,000.00
0[FV]	Set the face value	FV = 0.00
[CPT][PMT]	Compute the periodic payment	PMT = 3,334.99

Time Value of Money in Equity Instruments

Equity investments, such as stocks, enable an investor to acquire a fractional share/ownership by the issuing company. This gives investors the right to receive a share of the company's available cash flows as dividends.

In the context of equity instruments, the time value of money (TVM) is used to discount expected future cash flows to determine their present value. This allows investors to value the company shares.

The present value of expected future cash flows is calculated using a discount rate, r, which represents the expected rate of return on the investment.

Common Approaches for Valuing Equity Instruments

Valuing equity investments depends on the dividends cashflows which can take one of three forms: constant dividends, constant dividend growth rate, and changing dividend growth rate.

1. Valuing Equity Instruments based on Constant Dividend: The Constant Dividends model values stocks based on the assumption that dividends will remain constant over time. The preferred or common share dividend cash flows are in the form of an infinite series that is valued like perpetuity. The formula for the constant dividends model is as follows:

$$PV_t = \sum_{i=1}^{\infty} \frac{D_t}{(1+r)^i} = \frac{D_t}{r}$$

Where: PV_t = Present value at time t

D_t = Dividend payment at time t

r = Discount rate.

Example: Valuing Equity Instruments based on Constant Dividend

Assuming we have a preferred stock with a dividend payment of $5 per year. The discount rate is 8%. The present value of the stock is *closest to*:

Solution

Recall,

$$PV_t = \frac{D_t}{r}$$

In this case, D=$5, r=8%, PV=?

So,

$$PV = \frac{5}{0.08} = \$62.5$$

This means that the present value of the stock is $62.5.

2. Valuing Equity Instruments Based on Constant Dividend Growth Rate The constant dividend growth model is a method used to estimate the value of a stock based on its future dividends. This model assumes that dividends will grow at a constant rate (g) forever. To derive the formula for this model, we start by considering that the **present value of a stock is equal to the sum of its future dividends**, discounted by the required rate of return r. If dividends are assumed to grow at a constant rate, then each future dividend can be calculated by multiplying the previous dividend by $(1 + g)$.

Let D_t represent the expected dividend in the next period. The present value of the stock can then be expressed as:

$$PV_t = \frac{D_t}{(1+r)} + \frac{D_t(1+g)}{(1+r)^2} + \frac{D_t(1+g)^2}{(1+r)^3} + \ldots$$
$$= \sum_{i=1}^{\infty} \frac{D_t(1+g)}{(1+r)^i}$$

This is an infinite geometric series with a common ratio of $\frac{(1+r)}{(1+g)}$. Using the formula for the sum of an infinite geometric series, we can simplify this equation to:

$$PV_t = \frac{D_t(1+g)}{r-g} = \frac{D_{t+1}}{r-g}$$

Where:

PV_t = Present value at time t.

D_{t+1} = Expected Dividend in the next period.

r = Required rate of return.

g = Constant growth rate.

$r - g > 0$

Therefore, this is the formula for calculating the present value of a stock using the constant dividend growth rate. This model can help estimate the value of a stock when its future dividends are expected to grow steadily.

Example: Valuing Equity Instruments based on Constant Dividend Growth Rate

Suppose a stock currently pays an annual dividend of $2.00 per share. The required rate of return for this stock is 10%, and the dividends are expected to grow at a constant rate of 5% per year indefinitely. Using the constant dividend growth model, the present value of this stock is *closest to*:

Solution

Recall that,

$$PV_t = \frac{D_t(1+g)}{r-g} = \frac{D_{t+1}}{r-g}$$

In this case, we know that D_t=$2.00, r=10%, g=5%

So,

$$PV = \frac{2 \times 1.05}{r-g} = \frac{2.10}{0.10-0.05} = \$42$$

Therefore, the present value of the stock is $42

3. Valuing Equity Instruments with Changing Dividend Growth Rates is a dynamic process. It begins with the investor buying a stock at an initial price and getting an initial dividend. The unique aspect is that the dividend is expected to grow at a rate that evolves as the company matures and shifts from high growth to slower growth. This valuation doesn't have a single formula because it relies on assumptions about future dividend growth. However, a common method is to use a multi-stage dividend discount model. This model assumes that dividends will grow at different rates during various stages of the company's growth. To find the stock's present value, you sum up the present values of dividends at each stage.

The Multi-Stage Dividend Discount Model builds on the Constant Dividend Growth Model. It accommodates a company's transition from high initial growth to lower, more stable growth.

Let's say a company has a high short-term growth rate g_s followed by a perpetual lower growth rate g_l. To find the present value (PV) of the stock at time t using this model, we compute it in two stages:

I. **First Part**: The first part calculates the present value of dividends during the initial n periods of higher growth (g_s). This is done by discounting the dividends for each period by the required rate of return r using the following formula:

$$PV_t = \sum_{i=1}^{n} \frac{D_t(1 + g_s)^i}{(1 + r)^i}$$

Where:PV = Present value.n = Number of periods.D_t = Dividend at time (t).g_s = Initial higher dividend growth rate.r = Required rate of return.

II. **Second Part**: The second part calculates the present value of dividends after the initial n periods, assuming constant growth at a lower long-term rate (g_l). This can be simplified using the geometric series simplification, where $E(S_t + n)$ represents the terminal value or stock value in n periods:

$$PV_t = \frac{E(S_t + n)}{(1 + r)^n}$$

Where:$E(S_t + n) = \frac{D_{t+n+1}}{(r - g_l)}$ and g_l is the lower, more stable dividend growth rate.

Example: Valuing Equity Instruments based on Changing Dividend Growth Rate

Assuming we have a stock with an expected dividend payment of $2 in one period. The discount rate is 10%. The stock is expected to have a high dividend growth rate of 20% for the first three years, followed by a slower growth rate of 5% thereafter. Calculate the present value of the stock.

Solution

First, we calculate the present value of the dividends during the high growth period:

Recall that,

$$PV_t = \sum_{i=1}^{n} \frac{D_t(1 + g_s)^i}{(1 + r)^i}$$

In this case, D_t=\$2, g_s=0.20, r=0.10, n=3

So,

$$PV_1 = \frac{2}{(1+0.10)^1} + \frac{2 \times (1+0.20)^1}{1+0.10)^2} + \frac{2 \times (1+0.20)^2}{1+0.10)^3}$$
$$PV_1 = 1.818 + 1.983 + 2.163 = 5.965$$
$$PV_1 = \$5.97$$

Next, we calculate the present value of the dividends during the slower growth period, assuming that dividends will grow at a constant rate of 5% thereafter:

Recall that,

$$E(S_t + n) = \frac{D_{t+n+1}}{(r - g_l)}$$

So,

$$E(S_t + n) = \frac{2(1+0.20)^3 \times (1+0.05)}{0.10 - 0.05} = \frac{3.629}{0.05} = \$72.578$$

Finally, we calculate the present value of P_4

$$PV_2 = \frac{\$72.578}{(1+0.10)^3} = \$54.527$$

The total present value of the stock is the sum of PV_1 and PV_2

$$\begin{aligned} PV_{total} &= PV_1 + PV_2 \\ &= \$5.965 + 54.527 \\ &= \$60.493 \approx \$60.49 \end{aligned}$$