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Quantitative Methods, Economics, and Corporate Issuers

Level I Book 1

KAPLAN SCHWESER

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Again, thank you for trusting Kaplan Schweser with your CFA exam preparation. We're here to help you throughout your journey to become a CFA charterholder.

Regards,

Derek Burkett

Derek Burkett, CFA, FRM, CAIA
Vice President (Advanced Designations)

Book 1: Quantitative Methods, Economics, and Corporate Issuers

SchweserNotes™ 2025

Level I CFA®

KAPLAN SCHWESER

SCHWESERNOTES™ 2025 LEVEL I CFA® BOOK 1: QUANTITATIVE METHODS, ECONOMICS, AND CORPORATE ISSUERS

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CORPORATE ISSUERS

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   b. calculate and interpret different approaches to return measurement over time and describe their appropriate uses.

   c. compare the money-weighted and time-weighted rates of return and evaluate the performance of portfolios based on these measures.

   d. calculate and interpret annualized return measures and continuously compounded returns, and describe their appropriate uses.

   e. calculate and interpret major return measures and describe their appropriate uses.

2. **The Time Value of Money in Finance**

   The candidate should be able to:

   a. calculate and interpret the present value (PV) of fixed-income and equity instruments based on expected future cash flows.

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3. **Statistical Measures of Asset Returns**

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   a. calculate, interpret, and evaluate measures of central tendency and location to address an investment problem.

   b. calculate, interpret, and evaluate measures of dispersion to address an investment problem.

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4. **Probability Trees and Conditional Expectations**

   The candidate should be able to:

   a. calculate expected values, variances, and standard deviations and demonstrate their application to investment problems.

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   b. describe Monte Carlo simulation and explain how it can be used in investment applications.

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7. Estimation and Inference

The candidate should be able to:

a. compare and contrast simple random, stratified random, cluster, convenience, and judgmental sampling and their implications for sampling error in an investment problem.
b. explain the central limit theorem and its importance for the distribution and standard error of the sample mean.
c. describe the use of resampling (bootstrap, jackknife) to estimate the sampling distribution of a statistic.

8. Hypothesis Testing

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a. explain hypothesis testing and its components, including statistical significance, Type I and Type II errors, and the power of a test.
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9. Parametric and Non-Parametric Tests of Independence

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The candidate should be able to:

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e. calculate and interpret the predicted value for the dependent variable, and a prediction interval for it, given an estimated linear regression model and a value for the independent variable.
f. describe different functional forms of simple linear regressions.

11. Introduction to Big Data Techniques

The candidate should be able to:

a. describe aspects of "fintech" that are directly relevant for the gathering and analyzing of financial data.
b. describe Big Data, artificial intelligence, and machine learning.
c. describe applications of Big Data and Data Science to investment management.

12. Firms and Market Structures

The candidate should be able to:

a. determine and interpret breakeven and shutdown points of production, as well as how economies and diseconomies of scale affect costs under perfect and imperfect competition.
b. describe characteristics of perfect competition, monopolistic competition, oligopoly, and pure monopoly.
c. explain supply and demand relationships under monopolistic competition, including the optimal price and output for firms as well as pricing strategy.
d. explain supply and demand relationships under oligopoly, including the optimal price and output for firms as well as pricing strategy.

e. identify the type of market structure within which a firm operates and describe the use and limitations of concentration measures.

13. Understanding Business Cycles

The candidate should be able to:

a. describe the business cycle and its phases.
b. describe credit cycles.
c. describe how resource use, consumer and business activity, housing sector activity, and external trade sector activity vary over the business cycle and describe their measurement using economic indicators.

14. Fiscal Policy

The candidate should be able to:

a. compare monetary and fiscal policy.
b. describe roles and objectives of fiscal policy as well as arguments as to whether the size of a national debt relative to GDP matters.
c. describe tools of fiscal policy, including their advantages and disadvantages.
d. explain the implementation of fiscal policy and difficulties of implementation as well as whether a fiscal policy is expansionary or contractionary.

15. Monetary Policy

The candidate should be able to:

a. describe the roles and objectives of central banks.
b. describe tools used to implement monetary policy tools and the monetary transmission mechanism, and explain the relationships between monetary policy and economic growth, inflation, interest, and exchange rates.
c. describe qualities of effective central banks; contrast their use of inflation, interest rate, and exchange rate targeting in expansionary or contractionary monetary policy; and describe the limitations of monetary policy.
d. explain the interaction of monetary and fiscal policy.

16. Introduction to Geopolitics

The candidate should be able to:

a. describe geopolitics from a cooperation versus competition perspective.
b. describe geopolitics and its relationship with globalization.
c. describe functions and objectives of the international organizations that facilitate trade, including the World Bank, the International Monetary Fund, and the World Trade Organization.
d. describe geopolitical risk.
e. describe tools of geopolitics and their impact on regions and economies.
f. describe the impact of geopolitical risk on investments.

17. International Trade

The candidate should be able to:

a. describe the benefits and costs of international trade.
b. compare types of trade restrictions, such as tariffs, quotas, and export subsidies, and their economic implications.
c. explain motivations for and advantages of trading blocs, common markets, and economic unions.

18. Capital Flows and the FX Market

The candidate should be able to:

a. describe the foreign exchange market, including its functions and participants, distinguish between nominal and real exchange rates, and calculate and interpret the percentage change in a currency relative to another currency.
b. describe exchange rate regimes and explain the effects of exchange rates on countries' international trade and capital flows.
c. describe common objectives of capital restrictions imposed by governments.

19. Exchange Rate Calculations

The candidate should be able to:

a. calculate and interpret currency cross-rates.
b. explain the arbitrage relationship between spot and forward exchange rates and interest rates, calculate a forward rate using points or in percentage terms, and interpret a forward discount or premium.

20. Organizational Forms, Corporate Issuer Features, and Ownership

The candidate should be able to:

a. compare the organizational forms of businesses.
b. describe key features of corporate issuers.
c. compare publicly and privately owned corporate issuers.

21. Investors and Other Stakeholders

The candidate should be able to:

a. compare the financial claims and motivations of lenders and shareholders.
b. describe a company's stakeholder groups and compare their interests.
c. describe environmental, social, and governance factors of corporate issuers considered by investors.

22. Corporate Governance: Conflicts, Mechanisms, Risks, and Benefits

The candidate should be able to:

a. describe the principal-agent relationship and conflicts that may arise between stakeholder groups.
b. describe corporate governance and mechanisms to manage stakeholder relationships and mitigate associated risks.
c. describe potential risks of poor corporate governance and stakeholder management and benefits of effective corporate governance and stakeholder management.

23. Working Capital and Liquidity

The candidate should be able to:

a. explain the cash conversion cycle and compare issuers' cash conversion cycles.
b. explain liquidity and compare issuers' liquidity levels.
c. describe issuers' objectives and compare methods for managing working capital and liquidity.

24. Capital Investments and Capital Allocation

The candidate should be able to:

a. describe types of capital investments.
b. describe the capital allocation process, calculate net present value (NPV), internal rate of return (IRR), and return on invested capital (ROIC), and contrast their use in capital allocation.
c. describe principles of capital allocation and common capital allocation pitfalls.
d. describe types of real options relevant to capital investments.

25. Capital Structure

The candidate should be able to:

a. calculate and interpret the weighted-average cost of capital for a company.
b. explain factors affecting capital structure and the weighted-average cost of capital.
c. explain the Modigliani–Miller propositions regarding capital structure.
d. describe optimal and target capital structures.

26. Business Models

The candidate should be able to:

a. describe key features of business models.
b. describe various types of business models.

READING 1

RATES AND RETURNS

MODULE 1.1: INTEREST RATES AND RETURN MEASUREMENT

Video covering this content is available online.

LOS 1.a: Interpret interest rates as required rates of return, discount rates, or opportunity costs and explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk.

Interest rates measure the time value of money, although risk differences in financial securities lead to differences in their equilibrium interest rates. Equilibrium interest rates are the **required rate of return** for a particular investment, in the sense that the market rate of return is the return that investors and savers require to get them to willingly lend their funds. Interest rates are also referred to as **discount rates** and, in fact, the terms are often used interchangeably. If an individual can borrow funds at an interest rate of 10%, then that individual should discount payments to be made in the future at that rate to get their equivalent value in current dollars or other currencies. Finally, we can also view interest rates as the **opportunity cost** of current consumption. If the market rate of interest on 1-year securities is 5%, earning an additional 5% is the opportunity forgone when current consumption is chosen rather than saving (postponing consumption).

The **real risk-free rate** of interest is a theoretical rate on a single-period loan that contains no expectation of inflation and zero probability of default. What the real risk-free rate represents in economic terms is **time preference**, the degree to which current consumption is preferred to equal future consumption.

When we speak of a real rate of return, we are referring to an investor's increase in purchasing power (after adjusting for inflation). Because expected inflation in future periods is not zero, the rates we observe on U.S. Treasury bills (T-bills), for example, are essentially risk-free rates, but not real rates of return. T-bill rates are nominal risk-free rates because they contain an **inflation premium**. This is the relation:

$$(1 + \text{nominal risk-free rate}) = (1 + \text{real risk-free rate})(1 + \text{expected inflation rate})$$

Often, including in many parts of the CFA curriculum, this relation is approximated as follows:

$$\text{nominal risk-free rate} \approx \text{real risk-free rate} + \text{expected inflation rate}$$

Securities may have one or more types of risk, and each added risk increases the required rate of return. These types of risks are as follows:

- **Default risk.** This is the risk that a borrower will not make the promised payments in a timely manner.
- **Liquidity risk.** This is the risk of receiving less than fair value for an investment if it must be sold quickly for cash.
- **Maturity risk.** As we will see in the Fixed Income topic area, the prices of longer-term bonds are more volatile than those of shorter-term bonds. Longer-maturity bonds have more maturity risk than shorter-term bonds and require a maturity risk premium.

Each of these risk factors is associated with a risk premium that we add to the nominal risk-free rate to adjust for greater default risk, less liquidity, and longer maturity relative to a liquid, short-term, default risk-free rate such as that on T-bills. We can write the following:

$$\begin{aligned}\text{nominal rate of interest} = {} & \text{real risk-free rate} \\ & + \text{inflation premium} \\ & + \text{default risk premium} \\ & + \text{liquidity premium} \\ & + \text{maturity premium}\end{aligned}$$

LOS 1.b: Calculate and interpret different approaches to return measurement over time and describe their appropriate uses.

Holding period return (HPR) is simply the percentage increase in the value of an investment over a given period:

$$\text{holding period return} = \frac{\text{end-of-period value}}{\text{beginning-of-period value}} - 1$$

For example, a stock that pays a dividend during a holding period has an HPR for that period equal to:

$$\frac{P_t + Div_t}{P_0} - 1, \text{ or } \frac{P_t - P_0 + Div_t}{P_0}$$

If a stock is valued at €20 at the beginning of the period, pays €1 in dividends over the period, and at the end of the period is valued at €22, the HPR is:

$$\text{HPR} = (22 + 1) / 20 - 1 = 0.15 = 15\%$$

Returns over multiple periods reflect compounding. For example, given HPRs for Years 1, 2, and 3, the HPR for the entire three-year period is:

$$\text{HPR} = (1 + \text{HPR}_{\text{Year 1}})(1 + \text{HPR}_{\text{Year 2}})(1 + \text{HPR}_{\text{Year 3}}) - 1$$

Later in this reading, we will see that a return over multiple years is typically stated as an *annualized return* rather than an HPR.

Average Returns

The **arithmetic mean return** is the simple average of a series of periodic returns. It has the statistical property of being an unbiased estimator of the true mean of the underlying distribution of returns:

$$\text{arithmetic mean return} = \frac{(R_1 + R_2 + R_3 + \ldots + R_n)}{n}$$

The **geometric mean return** is a compound rate. When periodic rates of return vary from period to period, the geometric mean return will have a value less than the arithmetic mean return:

$$\text{geometric mean return} = \sqrt[n]{(1 + R_1) \times (1 + R_2) \times (1 + R_3) \times \ldots \times (1 + R_n)} - 1$$

For example, for returns R_t over three annual periods, the geometric mean return is calculated as the following example shows.

EXAMPLE: Geometric mean return

For the last three years, the returns for Acme Corporation common stock have been −9.34%, 23.45%, and 8.92%. Calculate the compound annual rate of return over the three-year period.

Answer:

$$R_G = \sqrt[3]{(1 - 0.0934) \times (1 + 0.2345) \times (1 + 0.0892)} - 1$$

$$= \sqrt[3]{0.9066 \times 1.2345 \times 1.0892} - 1$$

$$= \sqrt[3]{1.21903} - 1$$

$$R_G = 1.06825 - 1 = 6.825\%$$

Solve this type of problem with your calculator as follows:

- On the TI, enter 1.21903 [y^x] 3 [1/x] [=]
- On the HP, enter 1.21903 [ENTER] 3 [1/x] [y^x]

In the previous example, the geometric mean results in an annual rate of return because the holding periods were years. If the holding periods are other than years, the geometric mean is not the same as the annual return. The root for the geometric mean is the number of *periods*, while the root for the annual return is the number of *years*.

EXAMPLE: Geometric mean vs. annual return

For the last four semiannual periods, the 6-month holding period returns on an investment were 2.0%, 0.5%, −1.0%, and 1.5%. Calculate the geometric mean and the annual rate of return.

Answer:

$$\text{Geometric mean} = \sqrt[4]{(1+0.02)(1+0.005)(1-0.10)(1+0.015)} - 1$$
$$= 0.007435 = 0.7435\%.$$

This is the geometric mean of the 6-month holding period returns.

$$\text{Annual return} = \sqrt[2]{(1+0.02)(1+0.005)(1-0.10)(1+0.015)} - 1$$
$$= 0.0149 = 1.49\%.$$

The four semiannual periods equal two years, so to get an annual return we use 2 as the root.

PROFESSOR'S NOTE

The geometric mean is always less than or equal to the arithmetic mean, and the difference increases as the dispersion of the observations increases. The only time the arithmetic and geometric means are equal is when there is no variability in the observations (i.e., all observations are equal).

A **harmonic mean** is used for certain computations, such as the average cost of shares purchased over time. The harmonic mean is calculated as $\frac{N}{\sum_{i=1}^{N} \frac{1}{X_i}}$ where there are *N* values of X_i.

EXAMPLE: Calculating average cost with the harmonic mean

An investor purchases $1,000 of mutual fund shares each month, and over the last three months, the prices paid per share were $8, $9, and $10. What is the average cost per share?

Answer:

$$\overline{X}_H = \frac{3}{\frac{1}{8} + \frac{1}{9} + \frac{1}{10}} = \$8.926 \text{ per share}$$

To check this result, calculate the total shares purchased as follows:

$$\frac{1{,}000}{8} + \frac{1{,}000}{9} + \frac{1{,}000}{10} = 336.11 \text{ shares}$$

The average price is $\frac{\$3{,}000}{336.11} = \$8.926$ per share.

The previous example illustrates the interpretation of the harmonic mean in its most common application. Note that the average price paid per share ($8.93) is less than the arithmetic average of the share prices, which is $\frac{8+9+10}{3} = 9$.

We can only calculate a harmonic mean of positive numbers. For a set of returns that includes negative numbers, we can treat them the same way we did with geometric means, using (1 + return) for each period, then subtracting 1 from the result.

EXAMPLE: Harmonic mean with negative returns

For four periods, the returns on an investment were 2.0%, 0.5%, −1.0%, and 1.5%. Calculate the harmonic mean of these returns.

Answer:

$$\text{Harmonic mean} = \frac{4}{\frac{1}{(1+0.02)} + \frac{1}{(1+0.005)} + \frac{1}{(1-0.01)} + \frac{1}{(1+0.015)}} - 1$$

$$= 0.007369 = 0.7369\%$$

The relationship among arithmetic, geometric, and harmonic means can be stated as follows:

$$\text{arithmetic mean} \times \text{harmonic mean} = (\text{geometric mean})^2$$

PROFESSOR'S NOTE

The proof of this is beyond the scope of the Level I exam.

For values that are not all equal, harmonic mean < geometric mean < arithmetic mean. This mathematical fact is the basis for the claimed benefit of purchasing the same money amount of mutual fund shares each month or each week. Some refer to this practice as **cost averaging**.

Measures of average return can be affected by outliers, which are unusual observations in a dataset. Two of the methods for dealing with outliers are a *trimmed mean* and a *winsorized mean*. We will examine these in our reading on Statistical Measures of Asset Returns.

Appropriate uses for the various return measures are as follows:

- **Arithmetic mean.** Include all values, including outliers.
- **Geometric mean.** Compound the rate of returns over multiple periods.
- **Harmonic mean.** Calculate the average share cost from periodic purchases in a fixed money amount.
- **Trimmed or winsorized mean.** Decrease the effect of outliers.

MODULE QUIZ 1.1

1. An interest rate is *best* interpreted as a:
   A. discount rate or a measure of risk.
   B. measure of risk or a required rate of return.
   C. required rate of return or the opportunity cost of consumption.
2. An interest rate from which the inflation premium has been subtracted is known as a:
   A. real interest rate.
   B. risk-free interest rate.
   C. real risk-free interest rate.
3. The harmonic mean of 3, 4, and 5 is:
   A. 3.74.
   B. 3.83.

C. 4.12.

4. **XYZ Corp. Annual Stock Returns**

Year	20X1	20X2	20X3	20X4	20X5	20X6
Return	22%	5%	−7%	11%	2%	11%

The mean annual return on XYZ stock is *most appropriately* calculated using the:

A. harmonic mean.
B. arithmetic mean.
C. geometric mean.

MODULE 1.2: TIME-WEIGHTED AND MONEY-WEIGHTED RETURNS

Video covering this content is available online.

LOS 1.c: Compare the money-weighted and time-weighted rates of return and evaluate the performance of portfolios based on these measures.

The **money-weighted return** applies the concept of the **internal rate of return (IRR)** to investment portfolios. An IRR is the interest rate at which a series of cash inflows and outflows sum to zero when discounted to their present value. That is, they have a **net present value (NPV)** of zero. The IRR and NPV are built-in functions on financial calculators that CFA Institute permits candidates to use for the exam.

PROFESSOR'S NOTE

We have provided an online video in the Resource Library on how to use the TI calculator. You can view it by logging in to your account at www.schweser.com.

The **money-weighted rate of return** is defined as the IRR on a portfolio, taking into account all cash inflows and outflows. The beginning value of the account is an inflow, as are all deposits into the account. All withdrawals from the account are outflows, as is the ending value.

EXAMPLE: Money-weighted rate of return

Assume an investor buys a share of stock for $100 at t = 0, and at the end of the year (t = 1), she buys an additional share for $120. At the end of Year 2, the investor sells both shares for $130 each. At the end of each year in the holding period, the stock paid a $2 per share dividend. What is the money-weighted rate of return?

Step 1: Determine the timing of each cash flow and whether the cash flow is an inflow (+), into the account, or an outflow (−), available from the account.

$t = 0$:	purchase of first share	=	+$100.00	inflow to account
$t = 1$:	purchase of second share	=	+$120.00	
	dividends from first share	=	−$2.00	
	subtotal, $t = 1$		+$118.00	inflow to account
$t = 2$:	dividend from two shares	=	−$4.00	
	proceeds from selling shares	=	−$260.00	
	subtotal, $t = 2$		−$264.00	outflow from account

Step 2: Net the cash flows for each period and set the PV of cash inflows equal to the PV of cash outflows.

$$PV_{inflows} = PV_{outflows}$$

$$\$100 + \frac{\$118}{(1 + r)} = \frac{\$264}{(1 + r)^2}$$

Step 3: Solve for *r* to find the money-weighted rate of return. This can be done using trial and error or by using the IRR function on a financial calculator or spreadsheet.

The intuition here is that we deposited $100 into the account at t = 0, then added $118 to the account at t = 1 (which, with the $2 dividend, funded the purchase of one more share at $120), and ended with a total value of $264.

To compute this value with a financial calculator, use these net cash flows and follow the procedure(s) described to calculate the IRR:

$$\text{net cash flows: } CF_0 = +100;\ CF_1 = +120 - 2 = +118;$$
$$CF_2 = -260 + -4 = -264$$

Calculating money-weighted return with the TI Business Analyst II Plus®

Note the values for F01, F02, and so on, are all equal to 1.

Keystrokes	Explanation	Display
[CF] [2nd][CLR WORK]	Clear cash flow registers	CF0 = 0.00000
100 [ENTER]	Initial cash outlay	CF0 = +100.00000
[↓] 118 [ENTER]	Period 1 cash flow	C01 = +118.00000
[↓] [↓] 264 [+/−] [ENTER]	Period 2 cash flow	C02 = −264.00000
[IRR] [CPT]	Calculate IRR	IRR = 13.86122

The money-weighted rate of return for this problem is 13.86%.

PROFESSOR'S NOTE

In the preceding example, we entered the flows into the account as a positive and the ending value as a negative (the investor could withdraw this amount from the account). Note that there is no difference in the solution if we enter the cash flows into the account as negative values (out of the investor's pocket) and the ending value as a positive value (into the investor's pocket). As long as payments into the account and payments out of the account (including the ending value) are entered with opposite signs, the computed IRR will be correct.

Time-weighted rate of return measures compound growth and is the rate at which $1 compounds over a specified performance horizon. Time-weighting is the process of averaging a set of values over time. The annual time-weighted return for an investment may be computed by performing the following steps:

Step 1: Value the portfolio immediately preceding significant additions or withdrawals. Form subperiods over the evaluation period that correspond to the dates of deposits and withdrawals.

Step 2: Compute the holding period return (HPR) of the portfolio for each subperiod.

Step 3: Compute the product of (1 + HPR) for each subperiod to obtain a total return for the entire measurement period [i.e., $(1 + HPR_1) \times (1 + HPR_2) \ldots (1 + HPR_n)] - 1$.

If the total investment period is greater than one year, you must take the geometric mean of the measurement period return to find the annual time-weighted rate of return.

EXAMPLE: Time-weighted rate of return

An investor purchases a share of stock at $t = 0$ for $100. At the end of the year, $t = 1$, the investor buys another share of the same stock for $120. At the end of Year 2, the investor sells both shares for $130 each. At the end of both Years 1 and 2, the stock paid a $2 per share dividend. What is the annual time-weighted rate of return for this investment? (This is the same investment as the preceding example.)

Answer:

Step 1: Break the evaluation period into two subperiods based on timing of cash flows.

Holding period 1:	Beginning value	= $100
	Dividends paid	= $2
	Ending value	= $120
Holding period 2:	Beginning value	= $240 (2 shares)
	Dividends paid	= $4 ($2 per share)
	Ending value	= $260 (2 shares)

Step 2: Calculate the HPR for each holding period.

$$HPR_1 = [(\$120 + 2)/\$100] - 1 = 22\%$$

$$HPR_2 = [(\$260 + 4)/\$240] - 1 = 10\%$$

Step 3: Find the compound annual rate that would have produced a total return equal to the return on the account over the two-year period.

$$(1 + \text{time-weighted rate of return})^2 = (1.22)(1.10)$$

$$\text{time-weighted rate of return} = [(1.22)(1.10)]^{0.5} - 1 = 15.84\%$$

The time-weighted rate of return is not affected by the timing of cash inflows and outflows. In the investment management industry, time-weighted return is the preferred method of performance measurement because portfolio managers typically do not control the timing of deposits to and withdrawals from the accounts they manage.

In the preceding examples, the time-weighted rate of return for the portfolio was 15.84%, while the money-weighted rate of return for the same portfolio was 13.86%. The results are different because the money-weighted rate of return gave a larger weight to the Year 2 HPR, which was 10%, versus the 22% HPR for Year 1. This is because there was more money in the account at the beginning of the second period.

If funds are contributed to an investment portfolio just before a period of relatively poor portfolio performance, the money-weighted rate of return will tend to be lower than the time-weighted rate of return. On the other hand, if funds are contributed to a portfolio at a favorable time (just before a period of relatively high returns), the money-weighted rate of return will be higher than the time-weighted rate of return. The use of the time-weighted return removes these distortions, and thus provides a better measure of a manager's ability to select investments over the period. If the manager has complete control over money flows into and out of an account, the money-weighted rate of return would be the more appropriate performance measure.

MODULE QUIZ 1.2

1. An investor buys a share of stock for $40 at time $t = 0$, buys another share of the same stock for $50 at $t = 1$, and sells both shares for $60 each at $t = 2$. The stock paid a dividend of $1 per share at $t = 1$ and at $t = 2$. The periodic money-weighted rate of return on the investment is *closest* to:
   A. 22.2%.
   B. 23.0%.
   C. 23.8%.
2. An investor buys a share of stock for $40 at time $t = 0$, buys another share of the same stock for $50 at $t = 1$, and sells both shares for $60 each at $t = 2$. The stock paid a dividend of $1 per share at $t = 1$ and at $t = 2$. The time-weighted rate of return on the investment for the period is *closest* to:
   A. 24.7%.
   B. 25.7%.
   C. 26.8%.

MODULE 1.3: COMMON MEASURES OF RETURN

Video covering this content is available online.

LOS 1.d: Calculate and interpret annualized return measures and continuously compounded returns, and describe their appropriate uses.

Interest rates and market returns are typically stated as **annualized returns**, regardless of the actual length of the time period over which they occur. To annualize an HPR that is realized over a specific number of days, use the following formula:

$$\text{annualized return} = (1 + \text{HPR})^{365/\text{days}} - 1$$

EXAMPLE: Annualized return, shorter than one year

A saver deposits $100 into a bank account. After 90 days, the account balance is $100.75. What is the saver's annualized rate of return?

Answer:

$$\text{HPR} = \frac{100.75}{100} - 1 = 0.0075 = 0.75\%$$
$$\text{annualized return} = (1 + 0.0075)^{365/90} - 1 = 0.0308 = 3.08\%$$

EXAMPLE: Annualized return, longer than one year

An investor buys a 500-day government bill for $970 and redeems it at maturity for $1,000. What is the investor's annualized return?

Answer:

$$\text{HPR} = \frac{1000}{970} - 1 = 0.0309 = 3.09\%$$
$$\text{annualized return} = (1 + 0.0309)^{365/500} - 1 = 0.0225 = 2.25\%$$

In time value of money calculations (which we will address in more detail in our reading on The Time Value of Money in Finance), more frequent compounding has an impact on future value and present value computations. Specifically, because an increase in the frequency of compounding increases the effective interest rate, it also *increases* the future value of a given cash flow and *decreases* the present value of a given cash flow.

This is the general formula for the present value of a future cash flow:

$$\text{PV} = \text{FV}_N\left(1 + \frac{r}{m}\right)^{-mN}$$

where:
r = quoted annual interest rate
N = number of years
m = compounding periods per year

EXAMPLE: The effect of compounding frequency on FV and PV

Compute the PV of $1,000 to be received one year from now using a stated annual interest rate of 6% with a range of compounding periods.

Answer:

With semiannual compounding, m = 2:

$$\text{PV} = 1{,}000\left(1 + \frac{0.06}{2}\right)^{-2} = 942.60$$

With quarterly compounding, m = 4:

$$PV = 1{,}000\left(1 + \frac{0.06}{4}\right)^{-4} = 942.18$$

With monthly compounding, *m* = 12:

$$PV = 1{,}000\left(1 + \frac{0.06}{12}\right)^{-12} = 941.91$$

With daily compounding, *m* = 365:

$$PV = 1{,}000\left(1 + \frac{0.06}{365}\right)^{-365} = 941.77$$

Compounding Frequency Effect

Compounding Frequency	Interest Rate per Period	Present Value
Annual ($m = 1$)	6.000%	$943.40
Semiannual ($m = 2$)	3.000	942.60
Quarterly ($m = 4$)	1.500	942.18
Monthly ($m = 12$)	0.500	941.91
Daily ($m = 365$)	0.016438	941.77

The mathematical limit of shortening the compounding period is known as continuous compounding. Given an HPR, we can use the natural logarithm (ln, or LN on your financial calculator) to state its associated **continuously compounded return**:

$$R_{CC} = \ln(1 + HPR) = \ln\left(\frac{\text{ending value}}{\text{beginning value}}\right)$$

Notice that because the calculation is based on 1 plus the HPR, we can also perform it directly from the **price relative**. The price relative is just the end-of-period value divided by the beginning-of-period value.

EXAMPLE: **Calculating continuously compounded returns**

A stock was purchased for $100 and sold one year later for $120. Calculate the investor's annual rate of return on a continuously compounded basis.

Answer:

$$\ln\left(\frac{120}{100}\right) = 18.232\%$$

If we had been given the return (20%) instead, the calculation is this:

$$\ln(1 + 0.20) = 18.232\%$$

A useful property of continuously compounded rates of return is that they are additive for multiple periods. That is, a continuously compounded return from $t = 0$ to $t = 2$ is the sum of the continuously compounded return from $t = 0$ to $t = 1$ and the continuously compounded return from $t = 1$ to $t = 2$.

LOS 1.e: Calculate and interpret major return measures and describe their appropriate uses.

Gross return refers to the total return on a security portfolio before deducting fees for the management and administration of the investment account. **Net return** refers to the return after these fees have been deducted. Commissions on trades and other costs that are necessary to generate the investment returns are deducted in both gross and net return measures.

Pretax nominal return refers to the return before paying taxes. Dividend income, interest income, short-term capital gains, and long-term capital gains may all be taxed at different rates. **After-tax nominal return** refers to the return after the tax liability is deducted.

Real return is nominal return adjusted for inflation. Consider an investor who earns a nominal return of 7% over a year when inflation is 2%. The investor's approximate real return is simply 7 − 2 = 5%. The investor's exact real return is slightly lower: 1.07 / 1.02 − 1 = 0.049 = 4.9%.

Using the components of an interest rate we defined earlier, we can state a real return as follows:

$$(1 + \text{real return}) = \frac{(1 + \text{nominal risk-free rate})(1 + \text{risk premium})}{(1 + \text{inflation premium})}$$

PROFESSOR'S NOTE

The Level I curriculum states this relationship as

$$(1 + \text{real return}) = \frac{(1 + \text{real risk-free rate})(1 + \text{risk premium})}{(1 + \text{inflation premium})}.$$

Stating it this way assumes the risk premium includes inflation risk.

Real return measures the increase in an investor's purchasing power—how much more goods she can purchase at the end of one year due to the increase in the value of her investments. If she invests $1,000 and earns a nominal return of 7%, she will have $1,070 at the end of the year. If the price of the goods she consumes has gone up 2%, from $1.00 to $1.02, she will be able to consume 1,070 / 1.02 = 1,049 units. She has given up consuming 1,000 units today, but instead, she is able to purchase 1,049 units at the end of one year. Her purchasing power has gone up 4.9%; this is her real return.

A **leveraged return** refers to a return to an investor that is a multiple of the return on the underlying asset. The leveraged return is calculated as the gain or loss on the investment as a percentage of an investor's cash investment. An investment in a derivative security, such as a futures contract, produces a leveraged return because the cash deposited is only a fraction of the value of the assets underlying the futures contract. Leveraged investments in real estate are common: investors pay only a portion of a property's cost in cash and borrow the rest.

To illustrate the effect of leverage on returns, consider a fund that can invest the amount V_0 without leverage, and earn the rate of return *r*. The fund's unleveraged return (as a money amount) is simply $r \times V_0$. Now let's say this fund can borrow the amount V_B at an interest rate of r_B, and earn *r* by investing the proceeds. The fund's leveraged return (again as a money amount), after subtracting the interest cost, then becomes $r \times (V_0 + V_B) - (r_B \times V_B)$.

Thus, stated as a rate of return on the initial value of V_0, the leveraged rate of return is as follows:

$$\text{leveraged return} = \frac{r(V_0 + V_B) - r_B V_B}{V_0}$$

MODULE QUIZ 1.3

1. If an investment loses 3% of its value over 120 days, its annualized return is *closest* to:
   A. −8.0%.
   B. −8.5%.
   C. −9.0%.
2. If a stock's initial price is $20 and its price increases to $23, its continuously compounded rate of return is *closest* to:
   A. 13.64%.
   B. 13.98%.
   C. 15.00%.
3. The value of an investment increases 5% before commissions and fees. This 5% increase represents:
   A. the investment's net return.
   B. the investment's gross return.
   C. neither the investment's gross return nor its net return.

KEY CONCEPTS

LOS 1.a

An interest rate can be interpreted as the rate of return required in equilibrium for a particular investment, the discount rate for calculating the present value of future cash flows, or as the opportunity cost of consuming now, rather than saving and investing.

The real risk-free rate reflects time preference for present goods versus future goods. Nominal risk-free rate ≈ real risk-free rate + expected inflation rate.

Securities may have several risks, and each increases the required rate of return. These include default risk, liquidity risk, and maturity risk.

We can view a nominal interest rate as the sum of a real risk-free rate, expected inflation, a default risk premium, a liquidity premium, and a maturity premium.

LOS 1.b

Holding period return is used to measure an investment's return over a specific period. Arithmetic mean return is the simple average of a series of periodic returns. Geometric mean return is a compound annual rate.

Arithmetic mean return includes all observations, including outliers. Geometric mean return is used for compound returns over multiple periods. Harmonic mean is used to calculate the average price paid with equal periodic investments. Trimmed mean or winsorized mean are used to reduce the effect of outliers.

LOS 1.c

The money-weighted rate of return is the IRR calculated using periodic cash flows into and out of an account and is the discount rate that makes the PV of cash inflows equal to the PV of cash outflows.

The time-weighted rate of return measures compound growth and is the rate at which money compounds over a specified performance horizon.

If funds are added to a portfolio just before a period of poor performance, the money-weighted return will be lower than the time-weighted return. If funds are added just before a period of high returns, the money-weighted return will be higher than the time-weighted return.

The time-weighted return is the preferred measure of a manager's ability to select investments. If the manager controls the money flows into and out of an account, the money-weighted return is the more appropriate performance measure.

LOS 1.d

Interest rates and market returns are typically stated on an annualized basis:

$$\text{annualized return} = (1 + \text{HPR})^{365/\text{days}} - 1$$

Given a holding period return, this is the associated continuously compounded return:

$$R_{cc} = \ln(1 + \text{HPR}) = \ln\left(\frac{\text{ending value}}{\text{beginning value}}\right)$$

LOS 1.e

Gross return is the total return after deducting commissions on trades and other costs necessary to generate the returns, but before deducting fees for the management and administration of the investment account. Net return is the return after management and administration fees have been deducted.

Pretax nominal return is the numerical percentage return of an investment, without considering the effects of taxes and inflation. After-tax nominal return is the numerical return after the tax liability is deducted, without adjusting for inflation.

Real return is the increase in an investor's purchasing power, roughly equal to nominal return minus inflation.

Leveraged return is the gain or loss on an investment as a percentage of an investor's cash investment.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 1.1

1. **C** Interest rates can be interpreted as required rates of return, discount rates, or opportunity costs of current consumption. A risk premium can be, but is not always, a component of an interest rate. (LOS 1.a)

2. **A** Real interest rates are those that have been adjusted for inflation. (LOS 1.a)

3. **B** $\overline{X}_H = \dfrac{3}{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}} = 3.83$

   (LOS 1.b)

4. **C** Because returns are compounded, the geometric mean is appropriate.

   $[(1.22)(1.05)(0.93)(1.11)(1.02)(1.11)]^{1/6} - 1 = 6.96\%$

   (LOS 1.b)

Module Quiz 1.2

1. **C** Using the cash flow functions on your financial calculator, enter CF_0 = −40; CF_1 = −50 + 1 = −49; CF_2 = 60 ´ 2 + 2 = 122; and CPT IRR = 23.82%. (LOS 1.c)

2. **A** $HPR_1 = \dfrac{50+1}{40} - 1 = 27.5\%$

   $HPR_2 = \dfrac{120+2}{100} - 1 = 22.0\%$

   $TWR = \sqrt{(1+0.275)(1+0.22)} - 1 = 24.72\%$

   (LOS 1.c)

Module Quiz 1.3

1. **C** $\text{Annualized return} = (1-0.03)^{365/120} - 1 = -0.0885 = -8.85\%$

   (LOS 1.d)

2. **B** ln (23/20) = 0.1398

   (LOS 1.d)

3. **C** Gross return is the total return after deducting commissions on trades and other costs necessary to generate the returns, but before deducting fees for the management and administration of the investment account. Net return is the return after management and administration fees have been deducted. (LOS 1.e)

READING 2

THE TIME VALUE OF MONEY IN FINANCE

PROFESSOR'S NOTE

The examples we use in this reading are meant to show how the time value of money appears throughout finance. Don't worry if you are not yet familiar with the securities we describe in this reading. We will see these examples again when we cover bonds and forward interest rates in Fixed Income, stocks in Equity Investments, foreign exchange in Economics, and options in Derivatives.

WARM-UP: USING A FINANCIAL CALCULATOR

For the exam, you must be able to use a financial calculator when working time value of money problems. You simply do not have the time to solve these problems any other way.

CFA Institute allows only two types of calculators to be used for the exam: (1) the Texas Instruments® TI BA II Plus™ (including the BA II Plus Professional™) and (2) the HP® 12C (including the HP 12C Platinum). This reading is written primarily with the TI BA II Plus in mind. If you do not already own a calculator, purchase a TI BA II Plus! However, if you already own the HP 12C and are comfortable with it, by all means, continue to use it.

Before we begin working with financial calculators, you should familiarize yourself with your TI BA II Plus by locating the keys noted below. These are the only keys you need to know to calculate virtually all of the time value of money problems:

- N = number of compounding periods
- I/Y = interest rate per compounding period
- PV = present value
- FV = future value
- PMT = annuity payments, or constant periodic cash flow
- CPT = compute

The TI BA II Plus comes preloaded from the factory with the periods per year function (P/Y) set to 12. This automatically converts the annual interest rate (I/Y) into monthly rates. While appropriate for many loan-type problems, this feature is not suitable for the vast majority of the time value of money applications we will be studying. So, before using our SchweserNotes™, please set your P/Y key to "1" using the following sequence of keystrokes:

[2nd] [P/Y] "1" [ENTER] [2nd] [QUIT]

As long as you do not change the P/Y setting, it will remain set at one period per year until the battery from your calculator is removed (it does not change when you turn the calculator on and off). If you want to check this setting at any time, press [2nd] [P/Y]. The display should read P/Y = 1.0. If it does, press [2nd] [QUIT] to get out of the "programming" mode. If it does not, repeat the procedure previously described to set the P/Y key. With P/Y set to equal 1, it is now possible to think of I/Y as the interest rate per compounding period and *N* as the number of compounding periods under analysis. Thinking of these keys in this way should help you keep things straight as we work through time value of money problems.

PROFESSOR'S NOTE
We have provided an online video in the Resource Library on how to use the TI calculator. You can view it by logging in to your account at www.schweser.com.

MODULE 2.1: DISCOUNTED CASH FLOW VALUATION

Video covering this content is available online.

LOS 2.a: Calculate and interpret the present value (PV) of fixed-income and equity instruments based on expected future cash flows.

In our Rates and Returns reading, we gave examples of the relationship between present values and future values. We can simplify that relationship as follows:

$$FV = PV\,(1 + r)^t$$

$$PV = \frac{FV}{(1 + r)^t} = FV(1 + r)^{-t}$$

where:
r = interest rate per compounding period
t = number of compounding periods

If we are using continuous compounding, this is the relationship:

$$FV = PV \times e^{rt}$$

$$PV = FV \times e^{-rt}$$

Fixed-Income Securities

One of the simplest examples of the time value of money concept is a **pure discount** debt instrument, such as a **zero-coupon bond**. With a pure discount instrument, the investor pays less than the face value to buy the instrument and receives the face value at maturity. The price the investor pays depends on the instrument's **yield to maturity** (the discount rate applied to the face value) and the time until maturity. The amount of interest the investor earns is the difference between the face value and the purchase price.

> **EXAMPLE: Zero-coupon bond**
>
> A zero-coupon bond with a face value of $1,000 will mature 15 years from today. The bond has a yield to maturity of 4%. Assuming annual compounding, what is the bond's price?
>
> **Answer:**
>
> $$\text{PV} = \frac{\$1{,}000}{(1 + 0.04)^{15}} = \$555.26$$

We can infer a bond's yield from its price using the same relationship. Rather than solving for *r* with algebra, we typically use our financial calculators. For this example, if we were given the price of $555.26, the face value of $1,000, and annual compounding over 15 years, we would enter the following:

$\text{PV} = -555.26$

$\text{FV} = 1{,}000$

$\text{PMT} = 0$

$\text{N} = 15$

Then, to get the yield, CPT I/Y = 4.00.

PROFESSOR'S NOTE
Remember to enter cash outflows as negative values and cash inflows as positive values. From the investor's point of view, the purchase price (PV) is an outflow, and the return of the face value at maturity (FV) is an inflow.

In some circumstances, interest rates can be negative. A zero-coupon bond with a negative yield would be priced at a **premium**, which means its price is greater than its face value.

> **EXAMPLE: Zero-coupon bond with a negative yield**
>
> If the bond in the previous example has a yield to maturity of −0.5%, what is its price, assuming annual compounding?
>
> **Answer:**
>
> $$\text{PV} = \frac{\$1{,}000}{(1 - 0.005)^{15}} = \$1{,}078.09$$

A **fixed-coupon bond** is only slightly more complex. With a coupon bond, the investor receives a cash interest payment each period in addition to the face value at maturity. The bond's **coupon rate** is a percentage of the face value and determines the amount of the interest payments. For example, a 3% annual coupon, $1,000 bond pays 3% of $1,000, or $30, each year.

The coupon rate and the yield to maturity are two different things. We only use the coupon rate to determine the coupon payment (PMT). The yield to maturity (I/Y) is the discount rate implied by the bond's price.

EXAMPLE: Price of an annual coupon bond

Consider a 10-year, $1,000 par value, 10% coupon, annual-pay bond. What is the value of this bond if its yield to maturity is 8%?

Answer:

The coupon payments will be 10% × $1,000 = $100 at the end of each year. The $1,000 par value will be paid at the end of Year 10, along with the last coupon payment.

The value of this bond with a discount rate (yield to maturity) of 8% is:

$$\frac{100}{1.08}+\frac{100}{1.08^2}+\frac{100}{1.08^3}+\ldots+\frac{100}{1.08^9}+\frac{1{,}100}{1.08^{10}}=1{,}134.20$$

The calculator solution is:

$$\text{N}=10;\ \text{PMT}=100;\ \text{FV}=1{,}000;\ \text{I/Y}=8;\ \text{CPT PV}=-1{,}134.20$$

The bond's value is $1,134.20.

PROFESSOR'S NOTE

For this reading where we want to illustrate time value of money concepts, we are only using annual coupon payments and compounding periods. In the Fixed Income topic area, we will also perform these calculations for semiannual-pay bonds.

Some bonds exist that have no maturity date. We refer to these as **perpetual bonds** or **perpetuities**. We cannot speak meaningfully of the future value of a perpetuity, but its present value simplifies mathematically to the following:

$$\text{PV of a perpetuity}=\frac{\text{payment}}{\text{r}}$$

An **amortizing bond** is one that pays a level amount each period, including its maturity period. The difference between an amortizing bond and a fixed-coupon bond is that for an amortizing bond, each payment includes some portion of the principal. With a fixed-coupon bond, the entire principal is paid to the investor on the maturity date.

Amortizing bonds are an example of an **annuity** instrument. For an annuity, the payment each period is calculated as follows:

$$\text{annuity payment} = \frac{r \times PV}{1 - (1 + r)^{-t}}$$

where:

r = interest rate per period

t = number of periods

PV = present value (principal)

We can also determine an annuity payment using a financial calculator.

EXAMPLE: Computing a loan payment

Suppose you are considering applying for a $2,000 loan that will be repaid with equal end-of-year payments over the next 13 years. If the annual interest rate for the loan is 6%, how much are your payments?

Answer:

The size of the end-of-year loan payment can be determined by inputting values for the three known variables and computing PMT. Note that FV = 0 because the loan will be fully paid off after the last payment:

$$N = 13;\ I/Y = 6;\ PV = -2{,}000;\ FV = 0;\ CPT \rightarrow PMT = \$225.92$$

Equity Securities

As with fixed-income securities, we value **equity securities** such as common and preferred stock as the present value of their future cash flows. The key differences are that equity securities do not mature, and their cash flows may change over time.

Preferred stock pays a fixed dividend that is stated as a percentage of its **par value** (similar to the face value of a bond). As with bonds, we must distinguish between the stated percentage that determines the cash flows and the discount rate we apply to the cash flows. We say that equity investors have a **required return** that will induce them to own an equity share. This required return is the discount rate we use to value equity securities.

Because we can consider a preferred stock's fixed stream of dividends to be infinite, we can use the perpetuity formula to determine its value:

$$\text{preferred stock value} = \frac{D_p}{k_p}$$

where:

D_p = dividend per period

k_p = the market's required return on the preferred stock

EXAMPLE: Preferred stock valuation

A company's $100 par preferred stock pays a $5.00 annual dividend and has a required return of 8%. Calculate the value of the preferred stock.

Answer:

Value of the preferred stock: D_p/k_P = $5.00/0.08 = $62.50

Common stock is a residual claim to a company's assets after it satisfies all other claims. Common stock typically does not promise a fixed dividend payment. Instead, the company's management decides whether and when to pay common dividends.

Because the future cash flows are uncertain, we must use models to estimate the value of common stock. Here, we will look at three approaches analysts use frequently, which we call **dividend discount models (DDMs)**. We will return to these examples in the Equity Investments topic area and explain when each model is appropriate.

1. *Assume a constant future dividend*. Under this assumption, we can value a common stock the same way we value a preferred stock, using the perpetuity formula.
2. *Assume a constant growth rate of dividends*. With this assumption, we can apply the **constant growth DDM**, also known as the **Gordon growth model**. In this model, we state the value of a common share as follows:

$$V_0 = \frac{D_1}{k_e - g_c}$$

where:

V_0 = value of a share *this* period

D_1 = dividend expected to be paid *next* period

k_e = required return on common equity

g_c = constant growth rate of dividends

In this model, V_0 represents the PV of all the dividends in future periods, beginning with D_1. Note that k_e must be greater than g_c or the math will not work.

EXAMPLE: Gordon growth model valuation

Calculate the value of a stock that is expected to pay a $1.62 dividend next year, if dividends are expected to grow at 8% forever and the required return on equity is 12%.

Answer:

$$\text{Calculate the stock's value} = D_1/(k_e - g_c)$$

$$= \$1.62/(0.12 - 0.08)$$

$$= \$40.50$$

3. *Assume a changing growth rate of dividends*. This can be done in many ways. The example we will use here (and the one that is required for the Level I CFA exam) is known as a **multistage DDM**. Essentially, we assume a pattern of dividends in the short term, such as a period of high growth, followed by a constant growth rate of dividends in the long term.

   To use a multistage DDM, we discount the expected dividends in the short term as individual cash flows, then apply the constant growth DDM to the long term. As we saw in the previous example, the constant growth DDM gives us a value for an equity share *one period before* the dividend we use in the numerator.

EXAMPLE: Multistage growth

Consider a stock with dividends that are expected to grow at 15% per year for two years, after which they are expected to grow at 5% per year, indefinitely. The last dividend paid was $1.00, and k_e = 11%. Calculate the value of this stock using the multistage growth model.

Answer:

Calculate the dividends over the high growth period:

$$D_1 = D_0(1 + g^*) = 1.00(1.15) = \$1.15$$

$$D_2 = D_1(1 + g^*) = 1.15(1.15) = 1.15^2 = \$1.32$$

Calculate the first dividend of the constant-growth period:

$$D_3 = D_2(1 + g) = 1.32 \times 1.05 = \$1.386$$

Use the constant growth model to get P_2, a value for all the (infinite) dividends expected from time = 3 onward:

$$P_2 = \frac{D_3}{k_e - g_c} = \frac{1.386}{0.11 - 0.05} = \$23.10$$

Finally, we can sum the present values of dividends 1 and 2 and of P_2 to get the present value of all the expected future dividends during both the high-growth and constant-growth periods:

$$\frac{1.15}{1.11} + \frac{1.32 + 23.10}{(1.11)^2} = \$20.86$$

PROFESSOR'S NOTE

A key point to notice in this example is that when we applied the dividend in Period 3 to the constant growth model, it gave us a value for the stock in Period 2. To get a value for the stock today, we had to discount this value back by two periods, along with the dividend in Period 2 that was not included in the constant growth value.

MODULE QUIZ 2.1

1. Terry Corporation preferred stock is expected to pay a $9 annual dividend in perpetuity. If the required rate of return on an equivalent investment is 11%, one share of Terry preferred should be worth:
   A. $81.82.
   B. $99.00.
   C. $122.22.
2. Dover Company wants to issue a $10 million face value of 10-year bonds with an annual coupon rate of 5%. If the investors' required yield on Dover's bonds is 6%, the amount the company will receive when it issues these bonds (ignoring transactions costs) will be:
   A. less than $10 million.
   B. equal to $10 million.
   C. greater than $10 million.

MODULE 2.2: IMPLIED RETURNS AND CASH FLOW ADDITIVITY

Video covering this content is available online.

LOS 2.b: Calculate and interpret the implied return of fixed-income instruments and required return and implied growth of equity instruments given the present value (PV) and cash flows.

The examples we have seen so far illustrate the relationships among present value, future cash flows, and the required rate of return. We can easily rearrange these relationships and solve for the required rate of return, given a security's price and its future cash flows.

EXAMPLE: Rate of return for a pure discount bond

A zero-coupon bond with a face value of $1,000 will mature 15 years from today. The bond's price is $650. Assuming annual compounding, what is the investor's annualized return?

Answer:

$$\frac{\$1{,}000}{(1+r)^{15}} = \$650$$

$$(1+r)^{15} = \frac{\$1{,}000}{\$650} = 1.5385$$

$$r = 1.5385^{1/15} - 1 = 0.0291 = 2.91\%$$

EXAMPLE: Yield of an annual coupon bond

Consider the 10-year, $1,000 par value, 10% coupon, annual-pay bond we examined in an earlier example, when its price was $1,134.20 at a yield to maturity of 8%. What is its yield to maturity if its price decreases to $1,085.00?

Answer:

N = 10; PMT = 100; FV = 1,000; PV = −1,085; CPT I/Y = 8.6934

The bond's yield to maturity increased to 8.69%.

Notice that the relationship between prices and yields is inverse. *When the price decreases, the yield to maturity increases. When the price increases, the yield to maturity decreases.* Or, equivalently, *when the yield increases, the price decreases. When the yield decreases, the price increases*. We will use this concept again and again when we study bonds in the Fixed Income topic area.

In our examples for equity share values, we assumed the investor's required rate of return. In practice, the required rate of return on equity is not directly observable. Instead, we use share prices that we can observe in the market to derive implied required rates of return on equity, given our assumptions about their future cash flows.

For example, if we assume a constant rate of dividend growth, we can rearrange the constant growth DDM to solve for the required rate of return:

$$V_0 = \frac{D_1}{k_e - g_c}$$

$$k_e - g_c = \frac{D_1}{V_0}$$

$$k_e = \frac{D_1}{V_0} + g_c$$

That is, the required rate of return on equity is the ratio of the expected dividend to the current price (which we refer to as a share's **dividend yield**) plus the assumed constant growth rate.

We can also rearrange the model to solve for a stock's **implied growth rate**, given a required rate of return:

$$k_e = \frac{D_1}{V_0} + g_c$$

$$g_c = k_e - \frac{D_1}{V_0}$$

That is, the implied growth rate is the required rate of return minus the dividend yield.

LOS 2.c: Explain the cash flow additivity principle, its importance for the no-arbitrage condition, and its use in calculating implied forward interest rates,

forward exchange rates, and option values.

The **cash flow additivity principle** refers to the fact that the PV of any stream of cash flows equals the sum of the PVs of the cash flows. If we have two series of cash flows, the sum of the PVs of the two series is the same as the PVs of the two series taken together, adding cash flows that will be paid at the same point in time. We can also divide up a series of cash flows any way we like, and the PV of the "pieces" will equal the PV of the original series.

EXAMPLE: Cash flow additivity principle

A security will make the following payments at the end of the next four years: $100, $100, $400, and $100. Calculate the PV of these cash flows using the concept of the PV of an annuity when the appropriate discount rate is 10%.

Answer:

We can divide the cash flows so that we have:

$t = 1$	$t = 2$	$t = 3$	$t = 4$	
100	100	100	100	Cash flow series #1
0	0	300	0	Cash flow series #2
$100	$100	$400	$100	

The additivity principle tells us that to get the PV of the original series, we can just add the PVs of cash flow series #1 (a 4-period annuity) and cash flow series #2 (a single payment three periods from now).

For the annuity: N = 4; PMT = 100; FV = 0; I/Y = 10; CPT →
PV = −$316.99

For the single payment: N = 3; PMT = 0; FV = 300; I/Y = 10; CPT →
PV = −$225.39

The sum of these two values is 316.99 + 225.39 = $542.38.

The sum of these two (present) values is identical (except for rounding) to the sum of the present values of the payments of the original series:

$$\frac{100}{1.1} + \frac{100}{1.1^2} + \frac{400}{1.1^3} + \frac{100}{1.1^4} = \$542.38$$

This is a simple example of **replication**. In effect, we created the equivalent of the given series of uneven cash flows by combining a 4-year annuity of 100 with a 3-year zero-coupon bond of 300.

We rely on the cash flow additivity principle in many of the pricing models we see in the Level I CFA curriculum. It is the basis for the **no-arbitrage principle**, or "law of one price," which says that if two sets of future cash flows are identical under all conditions, they will have the same price today (or if they don't, investors will quickly buy the lower-priced one and sell the higher-priced one, which will drive their prices together).

Three examples of valuation based on the no-arbitrage condition are forward interest rates, forward exchange rates, and option pricing using a binomial model. We will explain each of these examples in greater detail when we address the related concepts in the Fixed Income, Economics, and Derivatives topic areas. For now, just focus on how they apply the principle that equivalent future cash flows must have the same present value.

Forward Interest Rates

A *forward interest rate* is the interest rate for a loan to be made at some future date. The notation used must identify both the length of the loan and when in the future the money will be borrowed. Thus, *1y1y* is the rate for a 1-year loan to be made one year from now; *2y1y* is the rate for a 1-year loan to be made two years from now; *3y2y* is the 2-year forward rate three years from now; and so on.

By contrast, a *spot interest rate* is an interest rate for a loan to be made today. We will use the notation S_1 for a 1-year rate today, S_2 for a 2-year rate today, and so on.

The way the cash flow additivity principle applies here is that, for example, borrowing for three years at the 3-year spot rate, or borrowing for one-year periods in three successive years, should have the same cost today. This relation is illustrated as follows: $(1 + S_3)^3 = (1 + S_1)(1 + 1y1y)(1 + 2y1y)$.

In fact, any combination of spot and forward interest rates that cover the same time period should have the same cost. Using this idea, we can derive **implied forward rates** from spot rates that are observable in the fixed-income markets.

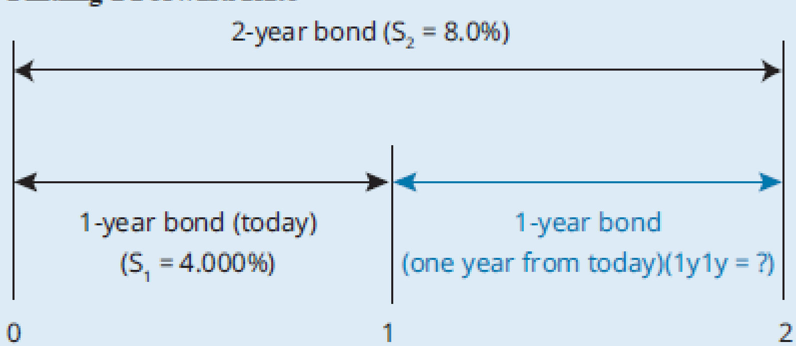
EXAMPLE: Computing a forward rate from spot rates

The 2-period spot rate, S_2, is 8%, and the 1-period spot rate, S_1, is 4%. Calculate the forward rate for one period, one period from now, *1y1y*.

Answer:

The following figure illustrates the problem.

Finding a Forward Rate



From our original equality, $(1 + S_2)^2 = (1 + S_1)(1 + 1y1y)$, we can get the following:

$$\frac{(1+S_2)^2}{(1+S_1)} = (1 + 1y1y)$$

Or, because we know that both choices have the same payoff in two years:

$$(1.08)^2 = (1.04)(1 + 1y1y)$$

$$(1 + 1y1y) = \frac{(1.08)^2}{(1.04)}$$

$$1y1y = \frac{(1.08)^2}{(1.04)} - 1 = \frac{1.1664}{1.04} - 1 = 12.154\%$$

In other words, investors are willing to accept 4.0% on the 1-year bond today (when they could get 8.0% on the 2-year bond today) only because they can get 12.154% on a 1-year bond one year from today. This future rate that can be locked in today is a forward rate.

Forward Currency Exchange Rates

An *exchange rate* is the price of one country's currency in terms of another country's currency. For example, an exchange rate of 1.416 USD/EUR means that one euro (EUR) is worth 1.416 U.S. dollars (USD). The Level I CFA curriculum refers to the currency in the numerator (USD, in this example) as the *price currency* and the one in the denominator (EUR in this example) as the *base currency*.

Like interest rates, exchange rates can be quoted as spot rates for currency exchanges to be made today, or as forward rates for currency exchanges to be made at a future date.

The percentage difference between forward and spot exchange rates is approximately the difference between the two countries' interest rates. This is because there is an arbitrage trade with a riskless profit to be made when this relation does not hold.

The possible arbitrage is as follows: borrow Currency A at Interest Rate A, convert it to Currency B at the spot rate and invest it to earn Interest Rate B, and sell the proceeds from this investment forward at the forward rate to turn it back into Currency A. If the forward rate does not correctly reflect the difference between interest rates, such an arbitrage could generate a profit to the extent that the return from investing Currency B and converting it back to Currency A with a forward contract is greater than the cost of borrowing Currency A for the period.

For spot and forward rates expressed as price currency/base currency, the no-arbitrage relation is as follows:

$$\frac{\text{forward}}{\text{spot}} = \frac{(1 + \text{interest rate}_{\text{price currency}})}{(1 + \text{interest rate}_{\text{base currency}})}$$

This formula can be rearranged as necessary to solve for specific values of the relevant terms.

EXAMPLE: Calculating the arbitrage-free forward exchange rate

Consider two currencies, the ABE and the DUB. The spot ABE/DUB exchange rate is 4.5671, the 1-year riskless ABE rate is 5%, and the 1-year riskless DUB rate is 3%. What is the 1-year forward exchange rate that will prevent arbitrage profits?

Answer:

Rearranging our formula, we have:

$$\text{forward} = \text{spot}\left(\frac{1 + I_{ABE}}{1 + I_{DUB}}\right)$$

and we can calculate the forward rate as:

$$\text{forward} = 4.5671\left(\frac{1.05}{1.03}\right) = 4.6558 \text{ ABE / DUB}$$

As you can see, the forward rate is greater than the spot rate by 4.6558 / 4.5671 − 1 = 1.94%. This is approximately equal to the interest rate differential of 5% − 3% = 2%.

Option Pricing Model

An *option* is the right, but not the obligation, to buy or sell an asset on a future date for a specified price. The right to buy an asset is a *call option*, and the right to sell an asset is a *put option*.

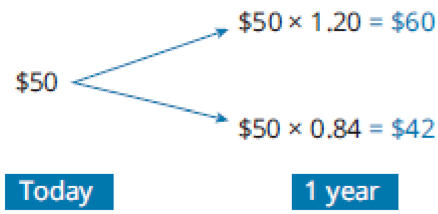
Valuing options is different from valuing other securities because the owner can let an option expire unexercised. A call option owner will let the option expire if the underlying asset can be bought in the market for less than the price specified in the option. A put option owner will let the option expire if the underlying asset can be sold in the market for more than the price specified in the option. In these cases, we say an option is *out of the money*. If an option is *in the money* on its expiration date, the owner has the right to buy the asset for less, or sell the asset for more, than its market price—and, therefore, will exercise the option.

An approach to valuing options that we will use in the Derivatives topic area is a **binomial model**. A binomial model is based on the idea that, over the next period, some value will change to one of two possible values. To construct a one-period binomial model for pricing an option, we need the following:

- A value for the underlying asset at the beginning of the period
- An exercise price for the option; the exercise price can be different from the value of the underlying, and we assume the option expires one period from now
- Returns that will result from an up-move and a down-move in the value of the underlying over one period
- The risk-free rate over the period

As an example, we can model a call option with an exercise price of $55 on a stock that is currently valued (S_0) at $50. Let us assume that in one period, the stock's value will either increase (S 1 u) to \$60 or decrease (S 1 d) to \$42. We state the return from an up-move (R^u) as \$60 / \$50 = 1.20, and the return from a down-move (R^d) as \$42 / \$50 = 0.84.

Figure 2.1: One-Period Binomial Tree



The call option will be in the money after an up-move or out of the money after a down-move. Its value at expiration after an up-move, c_1^u, is \$60 − \$55 = \$5. Its value after a down-move, c_1^d, is zero.

Now, we can use no-arbitrage pricing to determine the initial value of the call option (c_0). We do this by creating a portfolio of the option and the underlying stock, such that the portfolio will have the same value following either an up-move (V_1^u) or a down-move (V_1^d) in the stock. For our example, we would write the call option (that is, we grant someone else the option to buy the stock from us) and buy a number of shares of the stock that we will denote as *h*. We must solve for the *h* that results in $V_1^u = V_1^d$:

- The initial value of our portfolio, V_0, is $hS_0 - c_0$ (we subtract c_0 because we are short the call option).
- The portfolio value after an up-move, V_1^u, is $hS_1^u - c_1^u$.
- The portfolio value after a down-move, V_1^d, is $hS_1^d - c_1^d$.

In our example, $V_1^u = h(\$60) - \$5$, and $V_1^d = h(\$42) - 0$. Setting $V_1^u = V_1^d$ and solving for *h*, we get the following:

$$h(\$60) - \$5 = h(\$42)$$

$$h(\$60) - h(\$42) = \$5$$

$$h = \$5 \ / \ (\$60 - \$42) = 0.278$$

This result—the number of shares of the underlying we would buy for each call option we would write—is known as the hedge ratio for this option.

With $V_1^u = V_1^d$, the value of the portfolio after one period is known with certainty. This means we can say that either V_1^u or V_1^d must equal V_0 compounded at the risk-free rate for one period. In this example, V_1^d = 0.278(\$42) = \$11.68, or V_1^u = 0.278(\$60) − \$5 = \$11.68. Let us assume the risk-free rate over one period is 3%. Then, V_0 = \$11.68 / 1.03 = \$11.34.

Now, we can solve for the value of the call option, c_0. Recall that $V_0 = hS_0 - c_0$, so $c_0 = hS_0 - V_0$. Here, c_0 = 0.278(\$50) − \$11.34 = \$2.56.

MODULE QUIZ 2.2

1. For an equity share with a constant growth rate of dividends, we can estimate its:

A. value as the next dividend discounted at the required rate of return.
B. growth rate as the sum of its required rate of return and its dividend yield.
C. required return as the sum of its constant growth rate and its dividend yield.

2. An investment of €5 million today is expected to produce a one-time payoff of €7 million three years from today. The annual return on this investment, assuming annual compounding, is *closest* to:
A. 12%.
B. 13%.
C. 14%.

KEY CONCEPTS

LOS 2.a

The value of a fixed-income instrument or an equity security is the present value of its future cash flows, discounted at the investor's required rate of return:

$$PV = \frac{FV}{(1+r)^t} = FV(1+r)^{-t}$$

where:
r = interest rate per compounding period
t = number of compounding periods

$$\text{annuity payment} = \frac{r \times PV}{1-(1+r)^{-t}}$$

where:
r = interest rate per period
t = number of periods
PV = present value (principal)

The PV of a perpetual bond or a preferred stock $= \frac{\text{payment}}{r}$, where *r* = required rate of return.

The PV of a common stock with a constant growth rate of dividends is:

$$V_0 = \frac{D_1}{k_e - g_c}$$

LOS 2.b

By rearranging the present value relationship, we can calculate a security's required rate of return based on its price and its future cash flows. The relationship between prices and required rates of return is inverse.

For an equity share with a constant rate of dividend growth, we can estimate the required rate of return as the dividend yield plus the assumed constant growth rate, or we can estimate the implied growth rate as the required rate of return minus the dividend yield.

LOS 2.c

Using the cash flow additivity principle, we can divide up a series of cash flows any way we like, and the present value of the pieces will equal the present value of the original series. This principle is the basis for the no-arbitrage condition, under which two sets of future cash flows that are identical must have the same present value.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 2.1

1. **A** 9 / 0.11 = $81.82 (LOS 2.a)

2. **A** Because the required yield is greater than the coupon rate, the present value of the bonds is less than their face value: N = 10; I/Y = 6; PMT = 0.05 × $10,000,00 = $500,000; FV = $10,000,000; and CPT PV = −$9,263,991. (LOS 2.a)

Module Quiz 2.2

1. **C** Using the constant growth dividend discount model, we can estimate the required rate of return as $k_e = \frac{D_1}{V_0} + g_c$. The estimated value of a share is *all* of its future dividends discounted at the required rate of return, which simplifies to $V_0 = \frac{D_1}{k_e - g_c}$ if we assume a constant growth rate. We can estimate the constant growth rate as the required rate of return *minus* the dividend yield. (LOS 2.b)

2. **A** $\left(\frac{7}{5}\right)^{\frac{1}{3}} - 1 = 0.1187$

   (LOS 2.b)

READING 3

STATISTICAL MEASURES OF ASSET RETURNS

MODULE 3.1: CENTRAL TENDENCY AND DISPERSION

Video covering this content is available online.

LOS 3.a: Calculate, interpret, and evaluate measures of central tendency and location to address an investment problem.

Measures of Central Tendency

Measures of central tendency identify the center, or average, of a dataset. This central point can then be used to represent the typical, or expected, value in the dataset.

The **arithmetic mean** is the sum of the observation values divided by the number of observations. It is the most widely used measure of central tendency. An example of an arithmetic mean is a **sample mean**, which is the sum of all the values in a sample of a population, ΣX, divided by the number of observations in the sample, *n*. It is used to make *inferences* about the population mean. The sample mean is expressed as follows:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

The **median** is the midpoint of a dataset, where the data are arranged in ascending or descending order. Half of the observations lie above the median, and half are below. To determine the median, arrange the data from the highest to lowest value, or lowest to highest value, and find the middle observation.

The median is important because the arithmetic mean can be affected by **outliers**, which are extremely large or small values. When this occurs, the median is a better measure of central tendency than the mean because it is not affected by extreme values that may actually be the result of errors in the data.

EXAMPLE: The median using an odd number of observations

What is the median return for five portfolio managers with a 10-year annualized total returns record of 30%, 15%, 25%, 21%, and 23%?

Answer:

First, arrange the returns in descending order:

30%, 25%, 23%, 21%, 15%

Then, select the observation that has an equal number of observations above and below it—the one in the middle. For the given dataset, the third observation, 23%, is the median value.

EXAMPLE: The median using an even number of observations

Suppose we add a sixth manager to the previous example with a return of 28%. What is the median return?

Answer:

Arranging the returns in descending order gives us this:

30%, 28%, 25%, 23%, 21%, 15%

With an even number of observations, there is no single middle value. The median value, in this case, is the arithmetic mean of the two middle observations, 25% and 23%. Thus, the median return for the six managers is 24% = 0.5(25 + 23).

The **mode** is the value that occurs most frequently in a dataset. A dataset may have more than one mode, or even no mode. When a distribution has one value that appears most frequently, it is said to be **unimodal**. When a dataset has two or three values that occur most frequently, it is said to be **bimodal** or **trimodal**, respectively.

EXAMPLE: The mode

What is the mode of the following dataset?

Dataset: [30%, 28%, 25%, 23%, 28%, 15%, 5%]

Answer:

The mode is 28% because it is the value appearing most frequently.

For continuous data, such as investment returns, we typically do not identify a single outcome as the mode. Instead, we divide the relevant range of outcomes into intervals, and we identify the **modal interval** as the one into which the largest number of observations fall.

Methods for Dealing With Outliers

In some cases, a researcher may decide that outliers should be excluded from a measure of central tendency. One technique for doing so is to use a **trimmed mean**. A trimmed mean excludes a stated percentage of the most extreme observations. A 1% trimmed mean, for example, would discard the lowest 0.5% and the highest 0.5% of the observations.

Another technique is to use a **winsorized mean**. Instead of discarding the highest and lowest observations, we substitute a value for them. To calculate a 90% winsorized mean, for example, we would determine the 5th and 95th percentile of the observations, substitute the 5th percentile for any values lower than that, substitute the 95th percentile for any values higher than that, and then calculate the mean of the revised dataset. Percentiles are measures of location, which we will address next.

Measures of Location

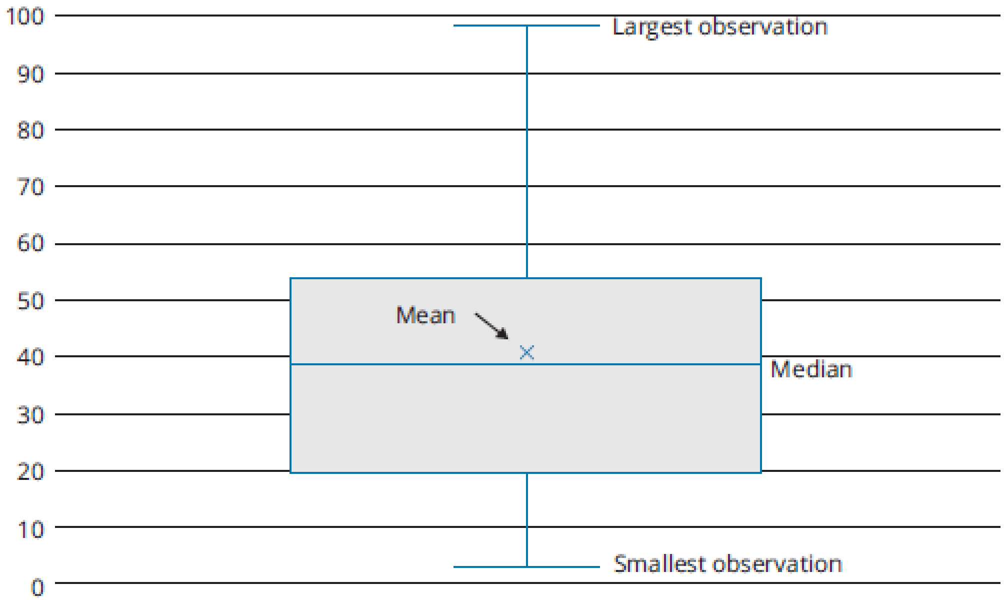
Quantile is the general term for a value at or below which a stated proportion of the data in a distribution lies. Examples of quantiles include the following:

- **Quartile.** The distribution is divided into quarters.
- **Quintile.** The distribution is divided into fifths.
- **Decile.** The distribution is divided into tenths.
- **Percentile.** The distribution is divided into hundredths (percentages).

Note that any quantile may be expressed as a percentile. For example, the third quartile partitions the distribution at a value such that three-fourths, or 75%, of the observations fall below that value. Thus, the third quartile is the 75th percentile. The difference between the third quartile and the first quartile (25th percentile) is known as the **interquartile range**.

To visualize a dataset based on quantiles, we can create a **box and whisker plot**, as shown in Figure 3.1. In a box and whisker plot, the box represents the central portion of the data, such as the interquartile range. The vertical line represents the entire range. In Figure 3.1, we can see that the largest observation is farther away from the center than is the smallest observation. This suggests that the data might include one or more outliers on the high side.

Figure 3.1: Box and Whisker Plot



LOS 3.b: Calculate, interpret, and evaluate measures of dispersion to address an investment problem.

Dispersion is defined as the *variability around the central tendency*. The common theme in finance and investments is the tradeoff between reward and variability, where the central tendency is the measure of the reward and dispersion is a measure of risk.

The **range** is a relatively simple measure of variability, but when used with other measures, it provides useful information. The range is the distance between the largest and the smallest value in the dataset:

range = maximum value − minimum value

> EXAMPLE: **The range**
>
> What is the range for the 5-year annualized total returns for five investment managers if the managers' individual returns were 30%, 12%, 25%, 20%, and 23%?
>
> **Answer:**
>
> range = 30 − 12 = 18%

The **mean absolute deviation (MAD)** is the average of the absolute values of the deviations of individual observations from the arithmetic mean:

$$\text{MAD} = \frac{\sum_{i=1}^{n} |X_i - \overline{X}|}{n}$$

The computation of the MAD uses the absolute values of each deviation from the mean because the sum of the actual deviations from the arithmetic mean is zero.

EXAMPLE: MAD

What is the MAD of the investment returns for the five managers discussed in the preceding example? How is it interpreted?

Answer:

Annualized returns: [30%, 12%, 25%, 20%, 23%]

$$\overline{X} = \frac{[30 + 12 + 25 + 20 + 23]}{5} = 22\%$$

$$MAD = \frac{[|30 - 22| + |12 - 22| + |25 - 22| + |20 - 22| + |23 - 22|]}{5}$$

$$MAD = \frac{[8 + 10 + 3 + 2 + 1]}{5} = 4.8\%$$

This result can be interpreted to mean that, on average, an individual return will deviate ±4.8% from the mean return of 22%.

The **sample variance**, s^2, is the measure of dispersion that applies when we are evaluating a sample of *n* observations from a population. The sample variance is calculated using the following formula:

$$s^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n - 1}$$

The denominator for s^2 is n − 1, one less than the sample size *n*. Based on the mathematical theory behind statistical procedures, the use of the entire number of sample observations, *n*, instead of n − 1 as the divisor in the computation of s^2, will systematically *underestimate* the population variance—particularly for small sample sizes. This systematic underestimation causes the sample variance to be a **biased estimator** of the population variance. Using n − 1 instead of *n* in the denominator, however, improves the statistical properties of s^2 as an estimator of the population variance.

EXAMPLE: Sample variance

Assume that the 5-year annualized total returns for the five investment managers used in the preceding examples represent only a sample of the managers at a large investment firm. What is the sample variance of these returns?

Answer:

$$\overline{X} = \frac{[30 + 12 + 25 + 20 + 23]}{5} = 22\%$$

$$s^2 = \frac{[(30 - 22)^2 + (12 - 22)^2 + (25 - 22)^2 + (20 - 22)^2 + (23 - 22)^2]}{5 - 1}$$

$$= 44.5(\%^2)$$

Thus, the sample variance of 44.5($\%^2$) can be interpreted to be an unbiased estimator of the population variance. Note that 44.5 "percent squared" is 0.00445, and you will get this value if you put the percentage returns in decimal form [e.g., $(0.30 - 0.22)^2$].

A major problem with using variance is the difficulty of interpreting it. The computed variance, unlike the mean, is in terms of squared units of measurement. How does one interpret squared percentages, squared dollars, or squared yen? This problem is mitigated through the use of the *standard deviation*. The units of standard deviation are the same as the units of the data (e.g., percentage return, dollars, euros). The **sample standard deviation** is the square root of the sample variance. The sample standard deviation, *s*, is calculated as follows:

$$s = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1}}$$

EXAMPLE: Sample standard deviation

Compute the sample standard deviation based on the result of the preceding example.

Answer:

Because the sample variance for the preceding example was computed to be 44.5($\%^2$), this is the sample standard deviation:

$$s = [44.5(\%^2)]^{1/2} = 6.67\%,\ \sqrt{0.00445} = 0.0667$$

This means that on average, an individual return from the sample will deviate ±6.67% from the mean return of 22%. The sample standard deviation can be interpreted as an unbiased estimator of the population standard deviation.

A direct comparison between two or more measures of dispersion may be difficult. For instance, suppose you are comparing the annual returns distribution for retail stocks with a mean of 8% and an annual returns distribution for a real estate portfolio with a mean of 16%. A direct comparison between the dispersion of the two distributions is not meaningful because of the relatively large difference in their means. To make a meaningful comparison, a relative measure of dispersion must be used. **Relative dispersion** is the amount of variability in a distribution around a reference point or benchmark. Relative dispersion is commonly measured with the **coefficient of variation (CV)**, which is computed as follows:

$$CV = \frac{s_x}{\overline{X}} = \frac{\text{standard deviation of x}}{\text{average value of x}}$$

CV measures the amount of dispersion in a distribution relative to the distribution's mean. This is useful because it enables us to compare dispersion across different sets of data. In an investments setting, the CV is used to measure the risk (variability) per unit of expected return (mean). A lower CV is better.

EXAMPLE: Coefficient of variation

You have just been presented with a report that indicates that the mean monthly return on T-bills is 0.25% with a standard deviation of 0.36%, and the mean monthly return for the S&P 500 is 1.09% with a standard deviation of 7.30%. Your manager has asked you to compute the CV for these two investments and to interpret your results.

Answer:

$$CV_{\text{T-bills}} = \frac{0.36}{0.25} = 1.44$$

$$CV_{\text{S\&P 500}} = \frac{7.30}{1.09} = 6.70$$

These results indicate that there is less dispersion (risk) per unit of monthly return for T-bills than for the S&P 500 (1.44 vs. 6.70).

PROFESSOR'S NOTE

To remember the formula for CV, remember that the CV is a measure of variation, so standard deviation goes in the numerator. CV is variation per unit of return.

When we use variance or standard deviation as risk measures, we calculate risk based on outcomes both above and below the mean. In some situations, it may be more appropriate to consider only outcomes less than the mean (or some other specific value) in calculating a risk measure. In this case, we are measuring **downside risk**.

One measure of downside risk is **target downside deviation**, which is also known as **target semideviation**. Calculating target downside deviation is similar to calculating standard deviation, but in this case, we choose a target value against which to measure each outcome and only include deviations from the target value in our calculation if the outcomes are below that target.

The formula for target downside deviation is stated as follows:

$$s_{\text{target}} = \sqrt{\frac{\sum_{\text{all } X_i < B}^{n} (X_i - B)^2}{n - 1}}$$

where B = the target

Note that the denominator remains the sample size *n* minus one, even though we are not using all of the observations in the numerator.

EXAMPLE: Target downside deviation

Calculate the target downside deviation based on the data in the preceding examples, for a target return equal to the mean (22%), and for a target return of 24%.

Answer:

Return	Deviation From Mean	Deviation From Target Return
30%	30% − 22% = 8%	30% − 24% = 6%
12%	12% − 22% = −10%	12% − 24% = −12%
25%	25% − 22% = 3%	25% − 24% = 1%
20%	20% − 22% = −2%	20% − 24% = −4%
23%	23% − 22% = 1%	23% − 24% = −1%

$$s_{22\%} = \sqrt{\frac{(-10)^2 + (-2)^2}{5-1}} = 5.10\%$$

$$s_{24\%} = \sqrt{\frac{(-12)^2 + (-4)^2 + (-1)^2}{5-1}} = 6.34\%$$

MODULE QUIZ 3.1

1. A dataset has 100 observations. Which of the following measures of central tendency will be calculated using a denominator of 100?
   A. The winsorized mean, but not the trimmed mean.
   B. Both the trimmed mean and the winsorized mean.
   C. Neither the trimmed mean nor the winsorized mean.
2. **XYZ Corp. Annual Stock Returns**

20X1	20X2	20X3	20X4	20X5	20X6
22%	5%	−7%	11%	2%	11%

   What is the sample standard deviation?
   A. 9.8%.
   B. 72.4%.
   C. 96.3%.
3. **XYZ Corp. Annual Stock Returns**

20X1	20X2	20X3	20X4	20X5	20X6
22%	5%	−7%	11%	2%	11%

   Assume an investor has a target return of 11% for XYZ stock. What is the stock's target downside deviation?
   A. 9.39%.
   B. 12.10%.
   C. 14.80%.

MODULE 3.2: SKEWNESS, KURTOSIS, AND CORRELATION

Video covering this content is available online.

LOS 3.c: Interpret and evaluate measures of skewness and kurtosis to address an investment problem.

A distribution is symmetrical if it is shaped identically on both sides of its mean. Distributional symmetry implies that intervals of losses and gains will exhibit the same frequency. For example, a symmetrical distribution with a mean return of zero will have losses in the −6% to −4% interval as frequently as it will have gains in the +4% to +6% interval. The extent to which a returns distribution is symmetrical is important because the degree of symmetry tells analysts if deviations from the mean are more likely to be positive or negative.

Skewness, or skew, refers to the extent to which a distribution is not symmetrical. Nonsymmetrical distributions may be either positively or negatively skewed and result from the occurrence of outliers in the dataset. **Outliers** are observations extraordinarily far from the mean, either above or below:

- A *positively skewed* distribution is characterized by outliers greater than the mean (in the upper region, or right tail). A positively skewed distribution is said to be skewed right because of its relatively long upper (right) tail.
- A *negatively skewed* distribution has a disproportionately large amount of outliers less than the mean that fall within its lower (left) tail. A negatively skewed distribution is said to be skewed left because of its long lower tail.

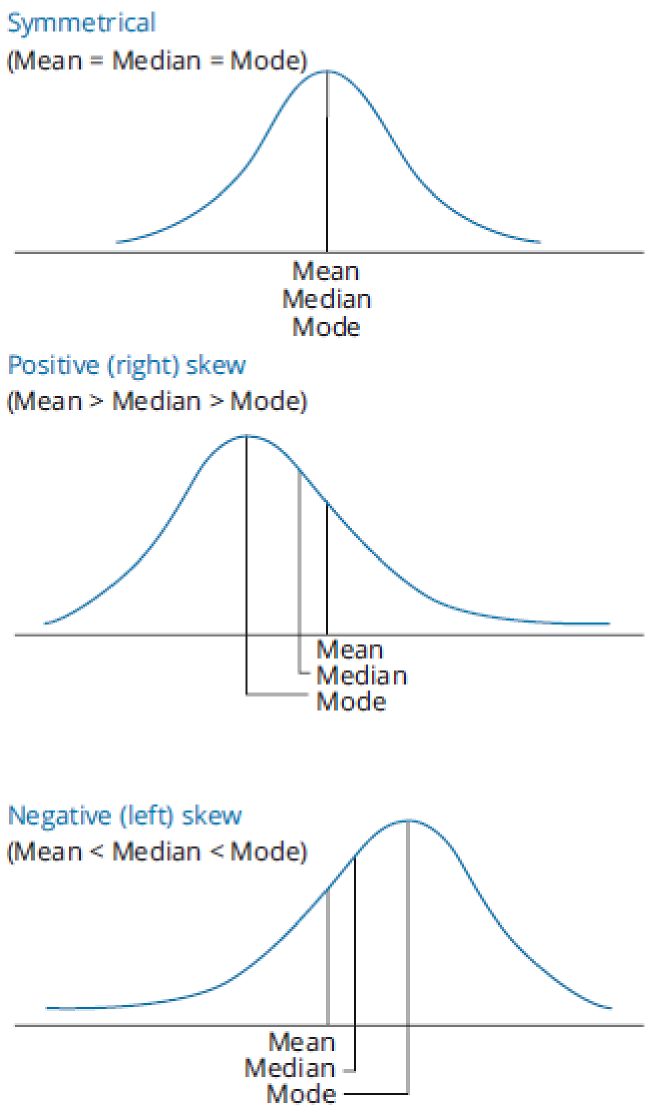
Skewness affects the location of the mean, median, and mode of a distribution:

- For a symmetrical distribution, the mean, median, and mode are equal.
- For a positively skewed, unimodal distribution, the mode is less than the median, which is less than the mean. The mean is affected by outliers; in a positively skewed distribution, there are large, positive outliers, which will tend to pull the mean upward, or more positive. An example of a positively skewed distribution is that of housing prices. Suppose you live in a neighborhood with 100 homes; 99 of them sell for $100,000, and one sells for $1,000,000. The median and the mode will be $100,000, but the mean will be $109,000. Hence, the mean has been pulled upward (to the right) by the existence of one home (outlier) in the neighborhood.
- For a negatively skewed, unimodal distribution, the mean is less than the median, which is less than the mode. In this case, there are large, negative outliers that tend to pull the mean downward (to the left).

PROFESSOR'S NOTE

The key to remembering how measures of central tendency are affected by skewed data is to recognize that skew affects the mean more than the median and mode, and the mean is pulled in the direction of the skew. The relative location of the mean, median, and mode for different distribution shapes is shown in Figure 3.2. Note that the median is between the other two measures for positively or negatively skewed distributions.

Figure 3.2: Effect of Skewness on Mean, Median, and Mode



Sample skewness is equal to the sum of the cubed deviations from the mean divided by the cubed standard deviation and by the number of observations. Sample skewness for large samples is approximated as follows:

$$\text{sample skewness} \approx \left(\frac{1}{n}\right)\frac{\sum_{i=1}^{n}(X_i - \overline{X})^3}{s^3}$$

where:

s = sample standard deviation

PROFESSOR'S NOTE
The LOS requires us to "interpret and evaluate" measures of skewness and kurtosis, but not to calculate them.

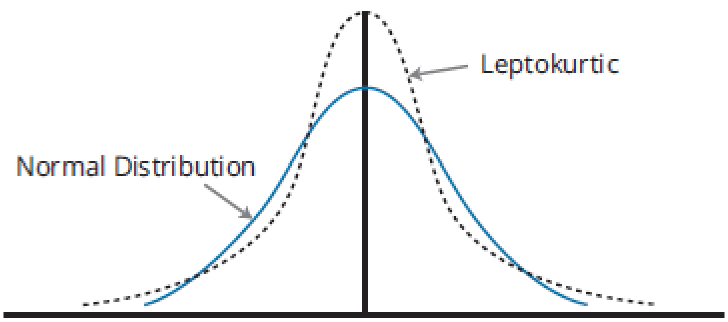
Note that the denominator is always positive, but that the numerator can be positive or negative depending on whether observations above the mean or observations below the mean tend to be farther from the mean, on average. When a distribution is right skewed, sample skewness is positive because the deviations above the mean are larger, on average. A left-skewed distribution has a negative sample skewness.

Dividing by standard deviation cubed standardizes the statistic and allows interpretation of the skewness measure. If relative skewness is equal to zero, the data are not skewed. Positive levels of relative skewness imply a positively skewed distribution, whereas negative values of relative skewness imply a negatively skewed distribution. Values of sample skewness in excess of 0.5 in absolute value are considered significant.

Kurtosis is a measure of the degree to which a distribution is more or less peaked than a normal distribution. **Leptokurtic** describes a distribution that is more peaked than a normal distribution, whereas **platykurtic** refers to a distribution that is less peaked, or flatter than a normal one. A distribution is **mesokurtic** if it has the same kurtosis as a normal distribution.

As indicated in Figure 3.3, a leptokurtic return distribution will have more returns clustered around the mean and more returns with large deviations from the mean (fatter tails). Relative to a normal distribution, a leptokurtic distribution will have a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean. This means that there is a relatively greater probability of an observed value being either close to the mean or far from the mean. Regarding an investment returns distribution, a greater likelihood of a large deviation from the mean return is often perceived as an increase in risk.

Figure 3.3: Kurtosis



A distribution is said to exhibit **excess kurtosis** if it has either more or less kurtosis than the normal distribution. The computed kurtosis for all normal distributions is three. Statisticians, however, sometimes report excess kurtosis, which is defined as kurtosis minus three. Thus, a normal distribution has excess kurtosis equal to zero, a leptokurtic distribution has excess kurtosis greater than zero, and platykurtic distributions will have excess kurtosis less than zero.

Kurtosis is critical in a risk management setting. Most research about the distribution of securities returns has shown that returns are not normally distributed. Actual securities returns tend to exhibit both skewness and kurtosis. Skewness and kurtosis are critical concepts for risk management because when securities returns are modeled using an assumed normal distribution, the predictions from the models will not take into account the potential for extremely large, negative outcomes. In fact, most risk managers put very little emphasis on the mean and standard deviation of a distribution and focus more on the distribution of returns in the tails of the distribution—that is where the risk is. In general, greater excess kurtosis and more negative skew in returns distributions indicate increased risk.

Sample kurtosis for large samples is approximated using deviations raised to the *fourth power*:

$$\text{sample kurtosis} \approx \left(\frac{1}{n}\right)\frac{\sum_{i=1}^{n}(X_i - \overline{X})^4}{s^4}$$

where:

s = sample standard deviation

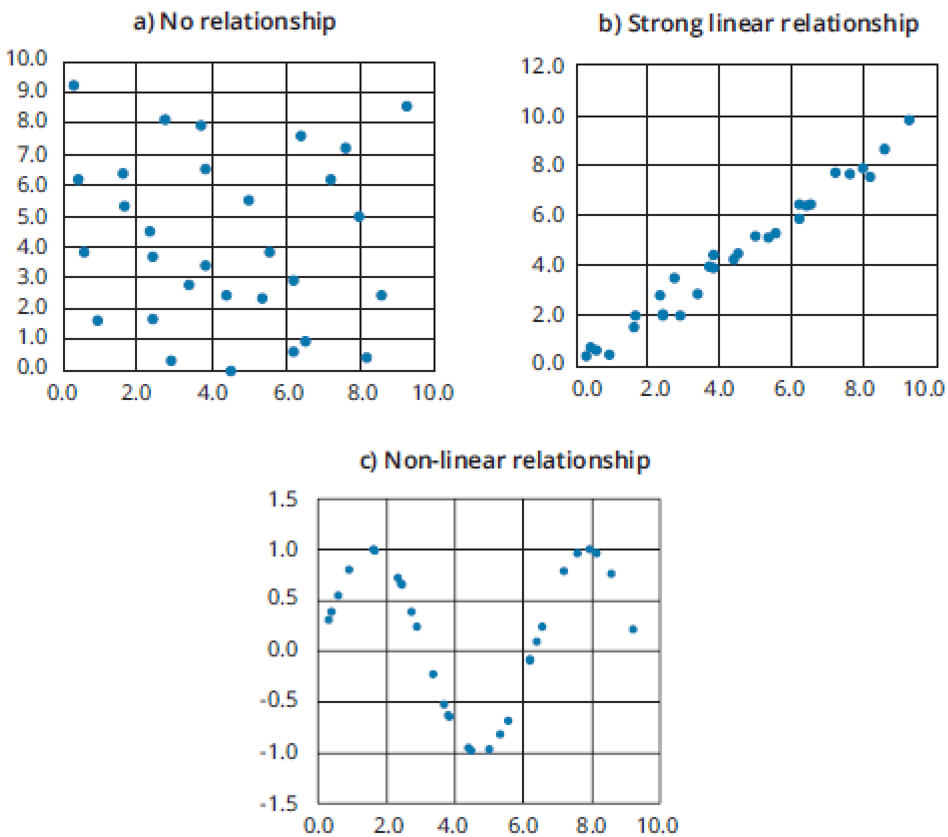
LOS 3.d: Interpret correlation between two variables to address an investment problem.

Scatter plots are a method for displaying the relationship between two variables. With one variable on the vertical axis and the other on the horizontal axis, their paired observations can each be plotted as a single point. For example, in Panel A of Figure 3.4, the point farthest to the upper right shows that when one of the variables (on the horizontal axis) equaled 9.2, the other variable (on the vertical axis) equaled 8.5.

The scatter plot in Panel A is typical of two variables that have no clear relationship. Panel B shows two variables that have a strong linear relationship—that is, a high correlation coefficient.

A key advantage of creating scatter plots is that they can reveal *nonlinear* relationships, which are not described by the correlation coefficient. Panel C illustrates such a relationship. Although the correlation coefficient for these two variables is close to zero, their scatter plot shows clearly that they are related in a predictable way.

Figure 3.4: Scatter Plots



Covariance is a measure of how two variables move together. The calculation of the **sample covariance** is based on the following formula:

$$s_{XY} = \frac{\sum_{i=1}^{n}\left\{\left[X_i - \overline{X}\right]\left[Y_i - \overline{Y}\right]\right\}}{n-1}$$

where:

X_i = an observation of variable X

Y_i = an observation of variable Y

$\overline{X}$ = mean of variable X

$\overline{Y}$ = mean of variable Y

n = number of periods

In practice, the covariance is difficult to interpret. The value of covariance depends on the units of the variables. The covariance of daily price changes of two securities priced in yen will be much greater than their covariance if the securities are priced in dollars. Like the variance, the units of covariance are the square of the units used for the data.

Additionally, we cannot interpret the relative strength of the relationship between two variables. Knowing that the covariance of *X* and *Y* is 0.8756 tells us only that they tend to move together because the covariance is positive. A standardized measure of the linear relationship between two variables is called the **correlation coefficient**, or simply *correlation*. The correlation between two variables, *X* and *Y*, is calculated as follows:

$\rho_{XY} = \frac{s_{XY}}{s_X s_Y}$, which implies:

$s_{XY} = \rho_{XY} s_X s_Y$

The *properties of the correlation* of two random variables, *X* and *Y*, are summarized here:

- Correlation measures the strength of the linear relationship between two random variables.
- Correlation has no units.
- The correlation ranges from −1 to +1. That is, $-1 \leq \rho_{XY} \leq +1$.
- If $\rho_{XY} = 1.0$, the random variables have perfect positive correlation. This means that a movement in one random variable results in a proportional positive movement in the other relative to its mean.
- If $\rho_{XY} = -1.0$, the random variables have perfect negative correlation. This means that a movement in one random variable results in an exact opposite proportional movement in the other relative to its mean.
- If $\rho_{XY} = 0$, there is no linear relationship between the variables, indicating that prediction of *Y* cannot be made on the basis of *X* using linear methods.

EXAMPLE: Correlation

The variance of returns on Stock A is 0.0028, the variance of returns on Stock B is 0.0124, and their covariance of returns is 0.0058. Calculate and interpret the correlation of the returns for Stocks A and B.

Answer:

First, it is necessary to convert the variances to standard deviations:

$$s_A = (0.0028)^{1/2} = 0.0529$$

$$s_B = (0.0124)^{1/2} = 0.1114$$

Now, the correlation between the returns of Stock A and Stock B can be computed as follows:

$$\rho_{AB} = \frac{0.0058}{(0.0529)(0.1114)} = 0.9842$$

The fact that this value is close to +1 indicates that the linear relationship is not only positive, but also is very strong.

Care should be taken when drawing conclusions based on correlation. Causation is not implied just from significant correlation. Even if it were, which variable is causing change in the other is not revealed by correlation. It is more prudent to say that two variables exhibit positive (or negative) association, suggesting that the nature of any causal relationship is to be separately investigated or based on theory that can be subject to additional tests.

One question that can be investigated is the role of outliers (extreme values) in the correlation of two variables. If removing the outliers significantly reduces the calculated correlation, further inquiry is necessary into whether the outliers provide information or are caused by noise (randomness) in the data used.

Spurious correlation refers to correlation that is either the result of chance or present due to changes in both variables over time that is caused by their association with a third variable. For example, we can find instances where two variables that are both related to the inflation rate exhibit significant correlation, but for which causation in either direction is not present.

In his book *Spurious Correlation*,[1] Tyler Vigen presents the following examples. The correlation between the age of each year's Miss America and the number of films Nicolas Cage appeared in that year is 87%. This seems a bit random. The correlation between the U.S. spending on science, space, and technology and suicides by hanging, strangulation, and suffocation over the 1999–2009 period is 99.87%. Impressive correlation, but both variables increased in an approximately linear fashion over the period.

MODULE QUIZ 3.2

1. Which of the following is *most accurate* regarding a distribution of returns that has a mean greater than its median?
   A. It is positively skewed.
   B. It is a symmetric distribution.
   C. It has positive excess kurtosis.
2. A distribution of returns that has a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean compared with a normal distribution:
   A. is positively skewed.
   B. has positive excess kurtosis.
   C. has negative excess kurtosis.
3. The correlation between two variables is +0.25. The *most appropriate* way to interpret this value is to say:
   A. a scatter plot of the two variables is likely to show a strong linear relationship.
   B. when one variable is above its mean, the other variable tends to be above its mean as well.
   C. a change in one of the variables usually causes the other variable to change in the same direction.

KEY CONCEPTS

LOS 3.a

The arithmetic mean is the average of observations. The sample mean is the arithmetic mean of a sample:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

The median is the midpoint of a dataset when the data are arranged from largest to smallest.