FRM Part 2

Book 1 – Market Risk Measurement and Management

THE ART OF TERM STRUCTURE MODELS: Drift

Learning Objectives

After completing this reading you should be able to:

- ✓ Construct and describe the effectiveness of a **short-term interest rate tree** assuming normally distributed rates, both **with and without drift**.
- ✓ Calculate the **short-term rate change** and **standard deviation** of the rate change using a model with normally distributed rates and no drift.
- ✓ Describe methods for addressing the **possibility of negative short-term rates** in term structure models.
- ✓ Construct a short-term rate tree under the **Ho-Lee Model** with time-dependent drift.
- ✓ Describe uses and benefits of the **arbitrage-free models** and assess the issue of **fitting models to market prices**.
- ✓ Describe the process of constructing a simple and recombining tree for a short-term rate under the **Vasicek Model with mean reversion**.
- ✓ Calculate the **Vasicek Model rate change**, **standard deviation** of the rate change, **expected rate** in T years, and *half-life*.
- ✓ Describe the **effectiveness** of the Vasicek Model.

Introduction – Terminologies

- **Short term rate models** are the models used to describe the evolution of short rates.
- **Short rates are spot rates**
  - The short rate (r_t) is the **continuously compounded, annualized** interest rate at which an entity can **borrow money** for an infinitesimally short period of time.

- If we measure the **movement of the interest rates**, we will conclude that it has a **mean**/average.
  - **Drift** is the rate at which the average **changes.**

Model 1 – Normally Distributed Rates Without Drift

- Model 1 is used in cases where there is **no drift** and interest rates are **normally distributed**.
- Under this model, the continuously compounded, instantaneous rate r_t is assumed to evolve according to the following equation:

$$dr = \sigma dw$$

- where:
  - *dr* = **change in interest rates** over small time interval, *dt*
  - *dt* = **small time interval** (measured in years) (e.g., one month = 1/12, 2 months = 2/12, and so forth)
  - σ = **annual basis-point volatility** of rate changes
  - *dw* = **normally distributed random variable** with mean 0 and standard deviation $\sqrt{dt}$

Model 1 – Normally Distributed Rates Without Drift

Example: Estimating the Change in Short-Term Rate

- Suppose that:
  - The current value of the **short-term rate is 5.26%,**
  - **Volatility** equals **115 basis points** per year, and that
  - The **time interval** under consideration is **one month**.
- Mathematically, r_0 = 5.26%; σ = 1.15%; and dt = 1/12.
- A month passes and the **random variable *dw***, with its zero mean and its standard deviation of $\sqrt{\frac{1}{12}}$ (or 0.2887), happens to take on a **value of 0.25**.
- Determine the **short-term rate after one month**.

Solution

- The change in the short-term rate is given by:
  - $dr = \sigma dw$
  - $= 1.15\% \times 0.25 = 0.2875\%$
- New short-term rate = 5.26% + 0.2875% = 5.55%
  - Since the short-term rate **started at 5.26%,** the short-term rate **after a month is 5.55%**.

Model 1 – Normally Distributed Rates Without Drift

Standard Deviation

- Since the **expected value of *dw* is zero**, it follows that the **expected change** in the rate, or the drift, **is zero**.
- Since the standard deviation of the normally distributed variable, *dw,* is $\sqrt{dt}$, the **standard deviation of the change in the rate** is:

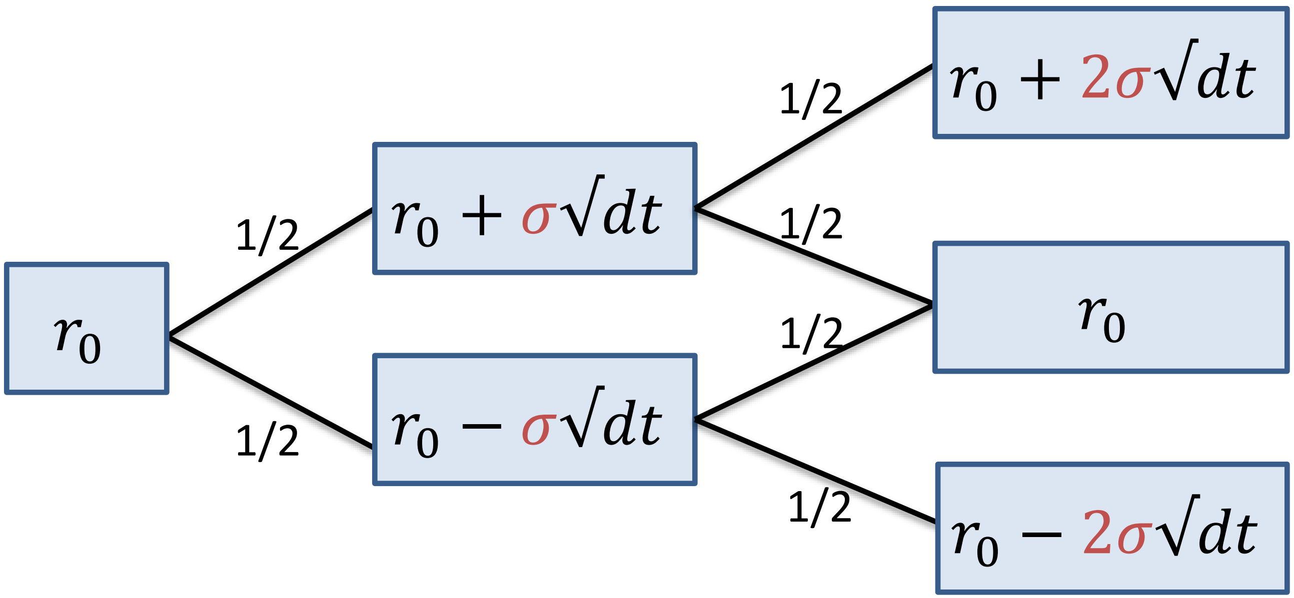
$$\sigma_{change\ in\ rate} = \sigma\sqrt{dt}$$

- For convenience the standard deviation of the rate of change is sometimes referred to as the **standard deviation of the rate**.

- In the previous example, the standard deviation of the rate is 1.15% × 0.2887 = 0.332% (or 33.2 basis points).

Model 1 – Interest Rate Tree Without Drift

- It is possible to build a zero drift interest rate tree using a binomial model.



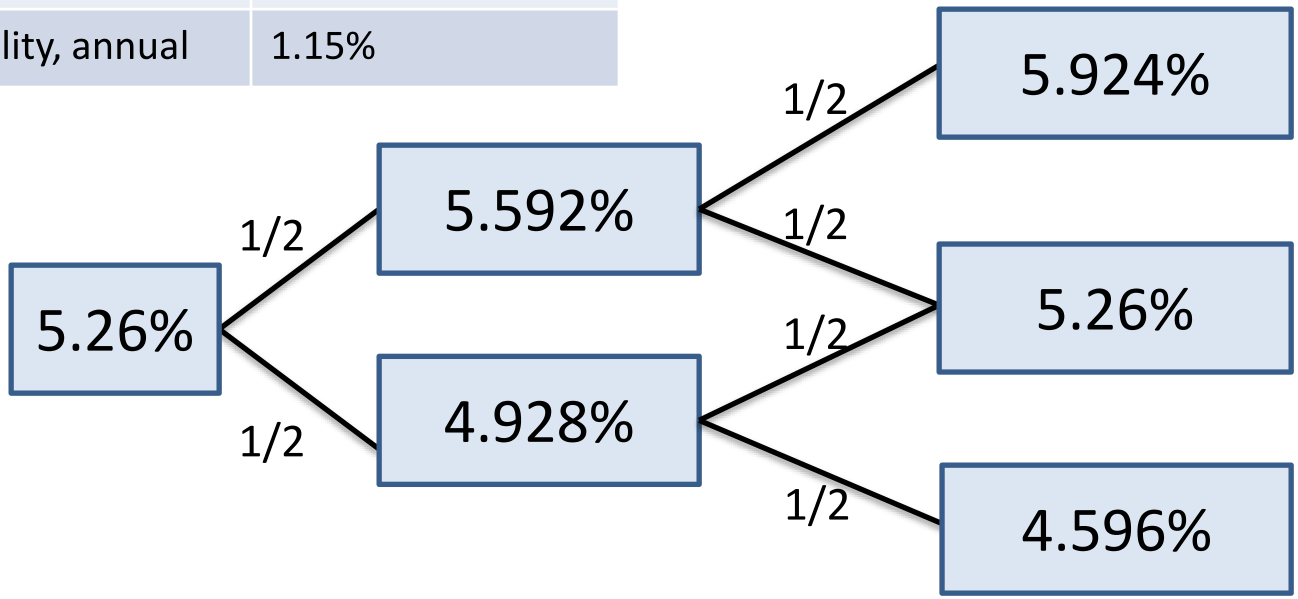
- Since drift is zero, rate **recombines** to current rate, r_0, at node [2,2].
- We can demonstrate how the expected change in the rate **(drift) is zero** as follows:

$$\text{Expected change in the rate} = \mathrm{E(dr)} = 0.5 \times \sigma\sqrt{\mathrm{dt}} + 0.5 \times -\sigma\sqrt{\mathrm{dt}} = 0$$

Model 1 – Interest Rate Tree Without Drift

Time	1/12
Initial short rate	5.26%
Drift, annual	N/A (zero)
Volatility, annual	1.15%

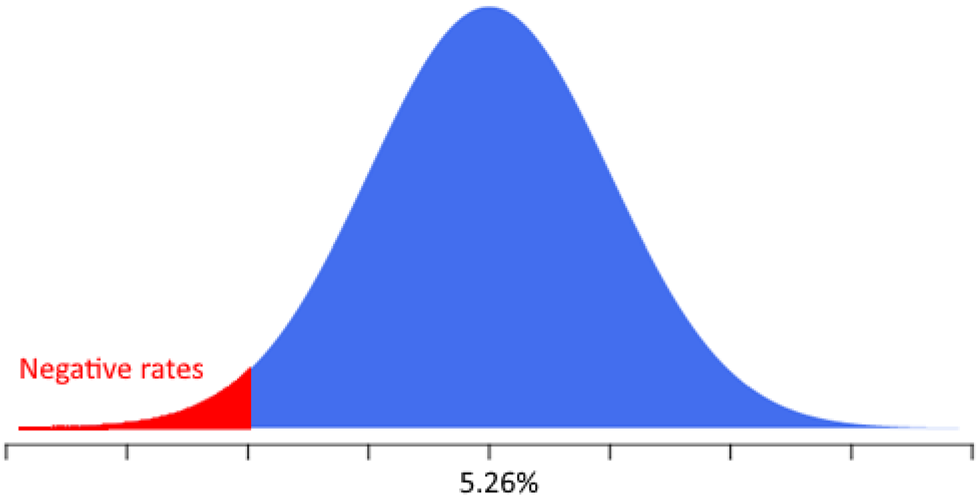
$$r_{t+1} = r_t + \sigma \sqrt{\frac{1}{12}}$$



- Top node in period 1 = 5.26% + 1.15% × 0.2887 = 5.592%
- Lower node in period 2 = 5.26% - (2 × 1.15% × 0.2887) = 4.596%

Possibility of Negative Short-Term Rates

- Model 1 poses a rather obvious scenario – the **possibility of negative interest rates**.
  - This is a problem with all models in which the terminal distribution of interest rates has a **normal distribution**.
- A negative interest rate makes **little economic sense**.
  - The possibility of a negative rate increases at the investment horizon gets longer since it is more likely that **forecasted interest rates will drop below zero**.



Methods for Addressing the Possibility of Negative Short-Term Rates

Changing the distribution of interest rates

- Possible candidates – **lognormal** or **chi-square**
- Downside: A model that locks out negative interest rates may pose problems in form of **skewness** and **inappropriate volatilities**

"Forcing" all negative rates to take a value of zero

- When there's a negative rate, the observed rate takes on a value of zero until the **shadow rate crosses back to a positive rate**.

Model 2 – Constant Drift

- Model 2 contains a **constant drift** and it is an extension to the Model 1.
  - The drift term is essentially a positive **risk premium associated with longer time horizons**.

Model 1	$dr = \sigma dw$
Model 2	$dr = \lambda dt + \sigma dw$

- Where
  - λ = drift
  - dt = **small time interval** (measured in years) (e.g., one month = 1/12, 2 months = 2/12, and so forth)
  - σ = **annual basis-point volatility** of rate changes
  - dw = **normally distributed random variable** with mean 0 and standard deviation $\sqrt{dt}$

Model 2 – Constant Drift

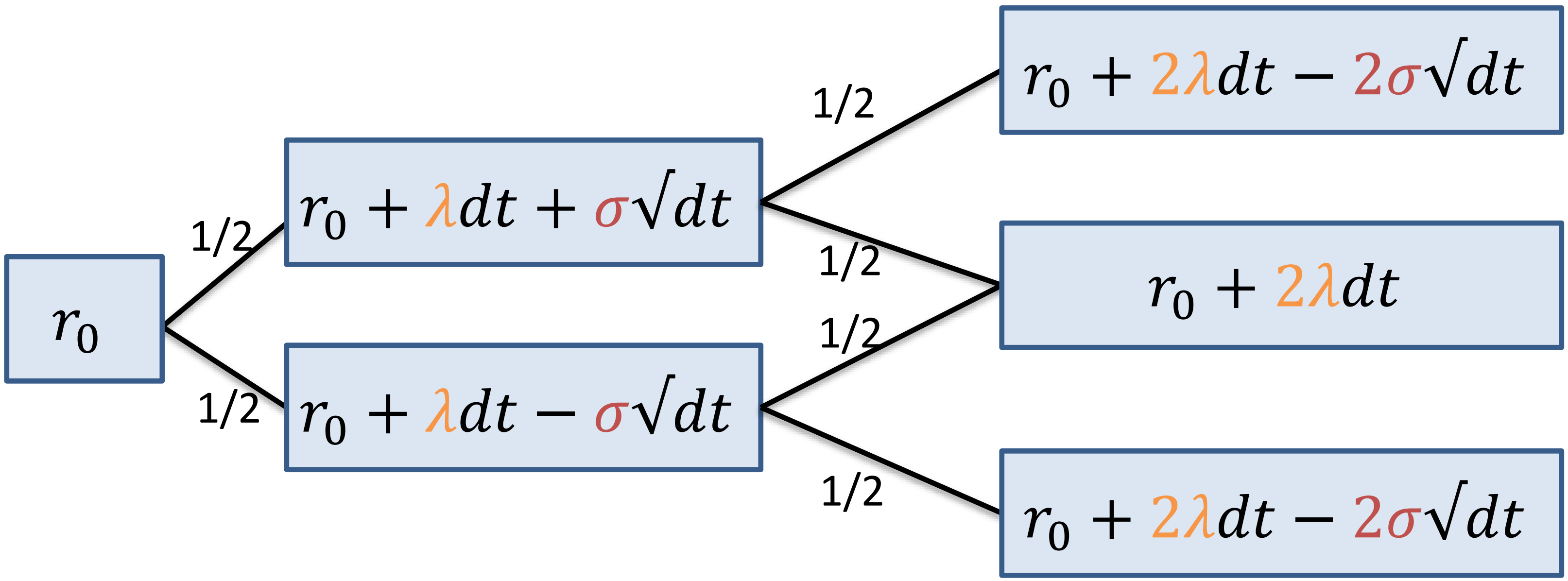
In our first example, we had:

- $dr = \sigma dw$
  - $= 1.15\% \times 0.25 = 0.2875\%$
- New short-term rate $= \ 5.26\% \ + \ 0.2875\% \ = \ \mathbf{5.55\%}$

If we now add an annual drift of 0.25%:

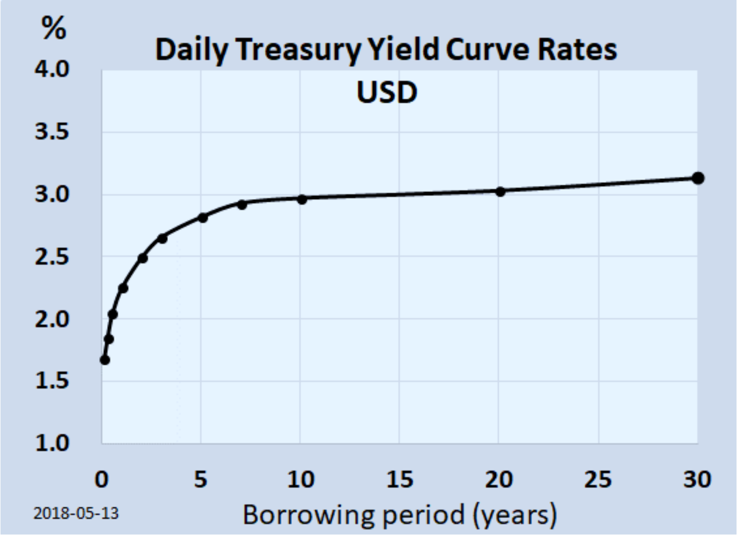
- The change in the short-term rate is given by:
  - $dr = \lambda dt + \sigma dw$
  - $= 0.25\% \times \frac{1}{12} + 1.15\% \times 0.25 = 0.3083\%$
- Since the short-term rate started at 5.26%, the short-term rate after a month is 5.5683%:
  - New short-term rate $= \ 5.26\% \ + \ 0.3083\% \ = \ 5.5683\%$
- The monthly drift is $0.25\% \ \times 1/12 \ = \ 0.0208\%$.
  - The 2.08 bps drift per month (0.0208%) represents any combination of expected changes in the short-term rate (i.e., **true drift**) and a **risk premium**.

2-period Interest Rate Tree with Drift (Model 2)



Which one is Better: Model 1 or Model 2?

- Model 2 is more effective than Model 1
  - Intuitively, the **drift term** accommodates the typically observed **upward-sloping nature** of the term structure.



- However, in the **long-term**, it is **difficult** to make a case for **rising expected rates**.

Ho-Lee Model

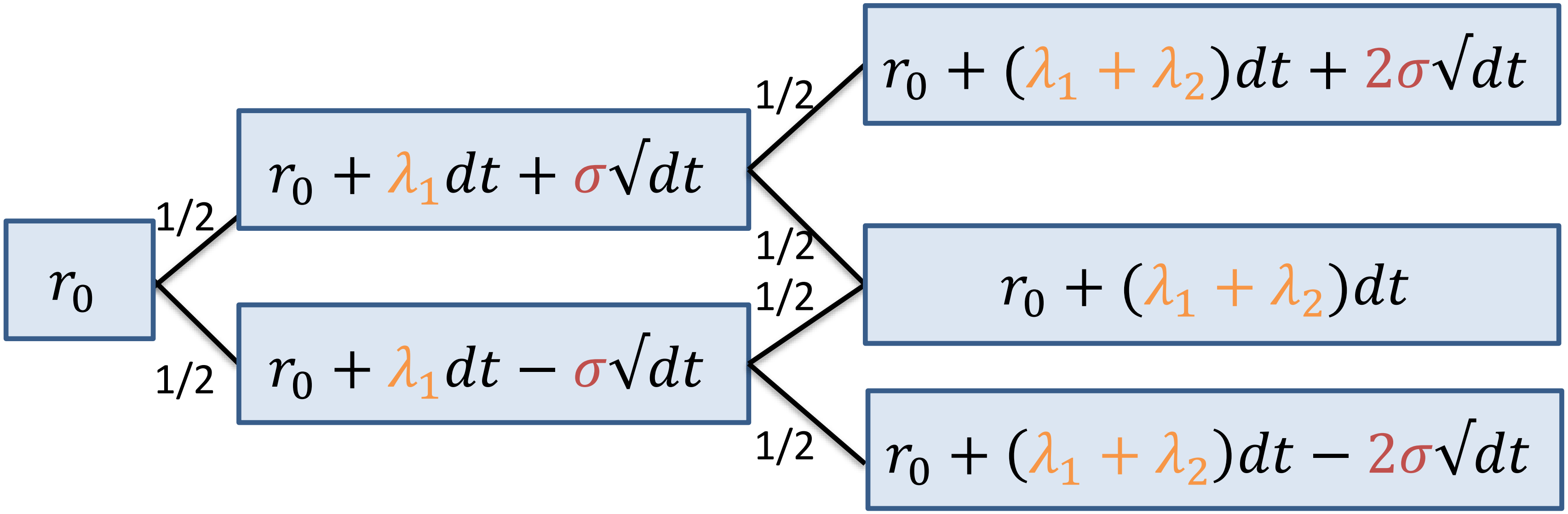
- Model 2 assumes that the drift (lambda) is **constant** from step to step along the tree.
  - The Ho-Lee Model assumes **that drift changes over time.**

Model 1	$dr = \sigma dw$
Model 2	$dr = \lambda dt + \sigma dw$
Ho-Lee Model	$dr = \lambda_t dt + \sigma dw$

- A drift that varies with time is called a ***time dependent*** **drift**.
  - For example, there might be an annualized drift of **10 basis points in month 1**, of **20 basis points in month 2**, and so on.

Interest rate tree >>

Interest Rate Tree under the Ho-Lee Model



- Where λ_1 and λ_1 are estimated from observed **market prices.**
  - The drift can move in **any** direction — it can be negative or positive for a given time interval.

Uses and Benefits of Arbitrage-Free Models

- In broad terms, there are two types of models: **arbitrage-free models** and **equilibrium models**.

__Arbitrage-free models__

- They are used to generate the true stochastic interest rate generating process by using real market data.
- They are particularly appropriate for **quoting the prices** of securities that do not have an active market traded based on the prices of more liquid securities (i.e., "**odd maturity**" swap with a term of four years and five months).

Disadvantages:

I. A **mediocre model cannot be rescued by calibrating it** to match market prices if, for example, the parallel shift assumption is not appropriate.

II. Arbitrage models assume the underlying prices are accurate **but** this will not be the case if there is an **external, temporary, exogenous shock**.

Equilibrium Short-Rate Models

- **Equilibrium short rate models** are based on the laws of economics such as supply-demand and require knowledge of **investor preferences** and **probabilities**.

- Unlike arbitrage-free short rate models, equilibrium short rate models make **certain assumptions about the true interest rate** generating process to determine the correct **theoretical term** structure.

- They are appropriate for **relative analysis** (i.e., comparing the value of two securities) particularly because they do not require the constraint that the underlying securities are priced accurately.
  - Arbitrage-free models are not appropriate for such an exercise because they **assume** that both securities are **correctly/properly priced**.

Vasicek Model (1/2)

- The Vasicek Model introduces **mean reversion** into the rate model.
  - When the **short-term rate is below** its long-run equilibrium value, the **drift is positive**, **driving the rate up** toward a long-run value.
  - And vice-versa.

Model 1	$dr = \sigma dw$
Model 2	$dr = \lambda dt + \sigma dw$
Ho-Lee Model	$dr = \lambda_t dt + \sigma dw$
Vasicek Model	$dr = k(\theta - r)dt + \sigma dw$

- Where:
  - k = a parameter that measures the speed of reversion adjustment
  - θ = long-run value of the short-term rate assuming risk neutrality
  - r = current interest rate level

Vasicek Model (2/2)

$$dr = k(\theta - r)dt + \sigma dw$$

- A **high *k*** will produce quicker (larger) **adjustments** than smaller values of *k*.
- Furthermore, the **greater** the **difference** between ***r*** **and *θ***, the **greater the expected change** in the short-term rate toward ***θ***.

- Under the assumption of risk-neutrality, the long-run value of the short-term rate can be approximated as:

$$\boldsymbol{\theta \approx r_l + \frac{\lambda}{k}}$$

  - Where r_l is the long-run true rate of interest.

Example >>

Non-Recombining Tree under the Vasicek Model (1/5)

- Representing a Vasicek interest rate process with a tree is not quite straightforward because it leads to a **non-recombining tree**.
- Let's demonstrate the process assuming a starting rate of 6%.

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

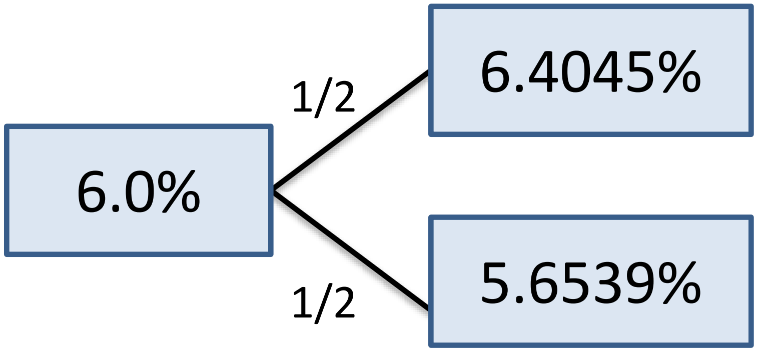
Non-Recombining Tree under the Vasicek Model (2/5)

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

1. First Period Upper and Lower Node Calculations

$$dr = k(\theta - r)dt \pm \sigma dw$$

Calculation



$$6.0\% + 0.05(13\% - 6\%)\left(\frac{1}{12}\right) + \frac{1.3\%}{\sqrt{12}}$$

$$6.0\% + 0.05(13\% - 6\%)\left(\frac{1}{12}\right) - \frac{1.3\%}{\sqrt{12}}$$

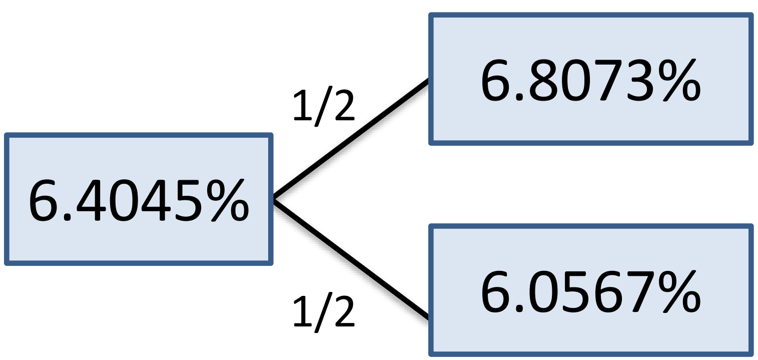
Non-Recombining Tree under the Vasicek Model (3/5)

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

2. Second Period Upper Node Calculations

$$dr = k(\theta - r)dt \pm \sigma dw$$

Calculation



$$6.4045\% + 0.05(13\% - 6.4045\%)\left(\frac{1}{12}\right) + \frac{1.3\%}{\sqrt{12}}$$

$$6.4045\% + 0.05(13\% - 6.4045\%)\left(\frac{1}{12}\right) - \frac{1.3\%}{\sqrt{12}}$$

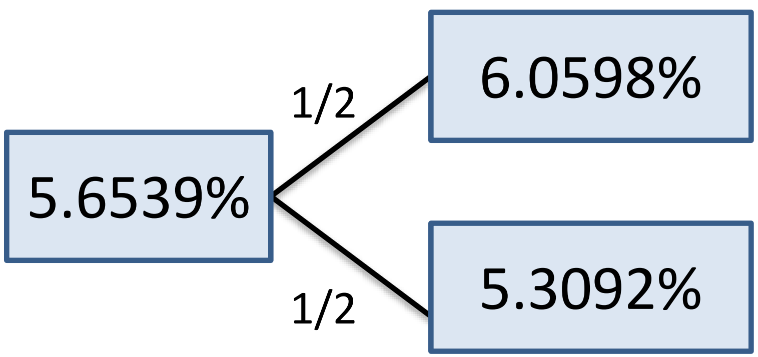
Non-Recombining Tree under the Vasicek Model (4/5)

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

3. Second Period Lower Node Calculations

$$dr = k(\theta - r)dt \pm \sigma dw$$

Calculation

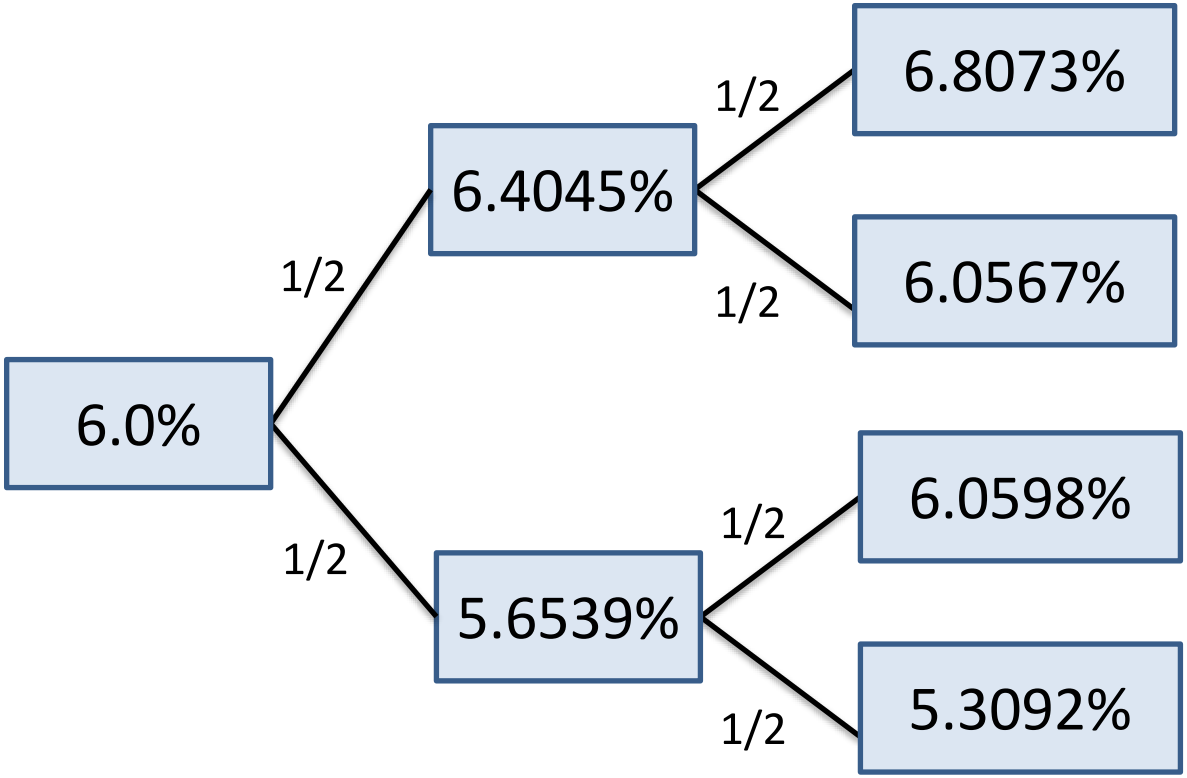


$$5.6539\% + 0.05(13\% - 5.6539\%)\left(\frac{1}{12}\right) + \frac{1.3\%}{\sqrt{12}}$$

$$5.6539\% + 0.05(13\% - 5.6539\%)\left(\frac{1}{12}\right) - \frac{1.3\%}{\sqrt{12}}$$

Non-Recombining Tree under the Vasicek Model (5/5)

Complete 2-Period Interest Rate (Non-combining) Tree with Mean Reversion

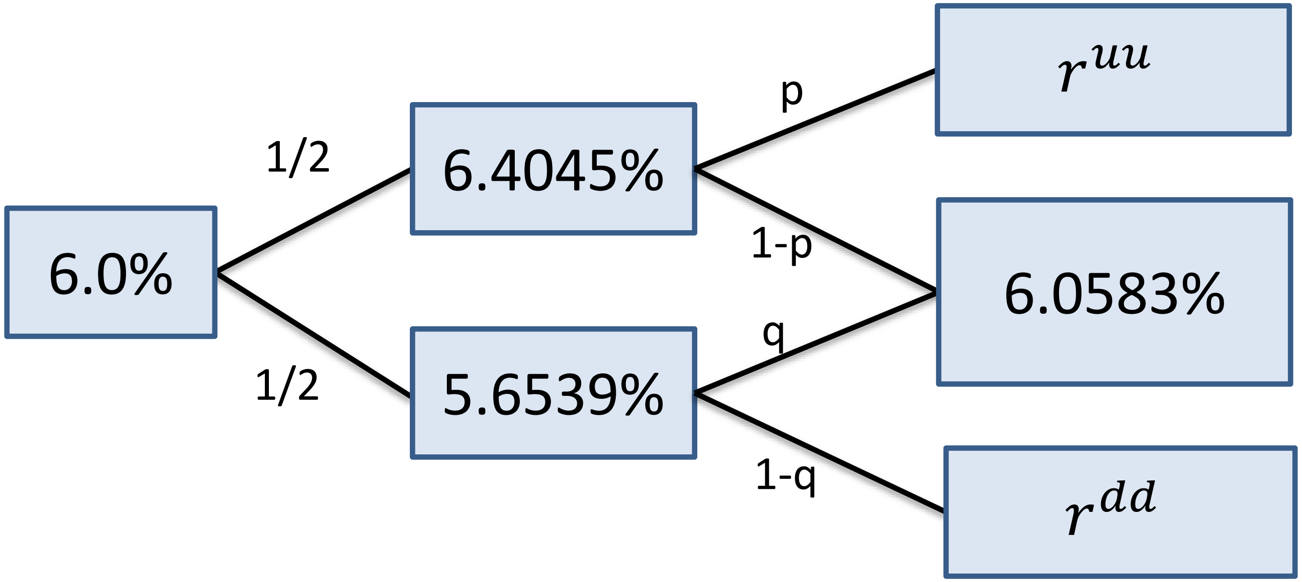


Recombining Tree under the Vasicek Model

Step 1: Find the **average** of the two middle **nodes**.

$$Average = \frac{6.0567\% + 6.0598\%}{2} = 6.0583\%$$

Step 2: Do away with the 50% probability of up-down movements and replace them with (p, 1 - p) and (q, 1 - q):



Step 3: Solve for p, q, r^{uu}, and r^{dd}.

Mean Reversion in the Vasicek Model

- The expectation (or mean) of the short-term rate as a function of horizon **gradually rises** from its current value toward its **limiting value of ϑ** – the long-run value of the short-term rate assuming risk neutrality.

Example

Given a current value of 6.0%, a long run value of 13% with the mean-reversion parameter, ***k*, of 0.05**, the long-term mean-reverting level will eventually reach 13%, but it will take a **long time** since the value of ***k* is quite small**.

The difference between the current value and the long-run value, i.e., 7%, decays **exponentially** at the rate of mean reversion.

Expected Rate in *T* Years and Half-Life

Expected Rate in T Years

- The expectation of the rate in the Vasicek model after *T* years is given by:

$$r_0 e^{-kT} + \theta\left(1 - e^{-kT}\right)$$

Half-Life

- The mean-reverting parameter *k* does not intuitively describe the pace of mean-reversion.
  - A more intuitive quantity is the factor's *half-life*, defined as the time it takes the factor to progress half the distance toward its goal.

$$\text{Half-life} = \tau_{years} = \frac{ln(2)}{k}$$

Effectiveness of the Vasicek Model

- The mean reversion parameter under the Vasicek model (1) improves the **specification of the term structure** and (2) produces a **specific term structure of volatility**.

- The Vasicek model will produce a term structure of volatility that is **declining**.
  - Particularly when we consider r_0 and θ calibrated to **match observed market prices**.

- As a result, the Vasicek model produces a term structure of volatility that is **declining**, implying that it **overstates short-term volatility** but **understates long-term volatility.**

- In contrast, Model 1 which has zero drift generates a **flat** volatility of interest rates across all maturities

Implication on Economic News

Mean reversion in the Vasicek model also has implications on long-lived and short-lived **economic news**.

i. Economic news is said to be ***long-lived*** if it changes the market's view of the economy many years into the future.
   - A good example would be the emergence of game-changing AI technology that raises productivity, triggering a long-lived shock to the system.

i. Economic news is said to be ***short-lived*** if it changes the market's view of the economy in the **near but not far future**.
   - A good example would be a **decline in retails sales** attributable to excessively cold weather **over the holiday season**.

Low mean rev. parameter = high half-life = long-lived effect on rates
High mean rev. parameter = short half-life = short-lived effect on rates

THE ART OF TERM STRUCTURE MODELS: Drift

Learning Objectives Recap:

- ✓ Construct and describe the effectiveness of a **short-term interest rate tree** assuming normally distributed rates, both **with and without drift**.
- ✓ Calculate the **short-term rate change** and **standard deviation** of the rate change using a model with normally distributed rates and no drift.
- ✓ Describe methods for addressing the **possibility of negative short-term rates** in term structure models.
- ✓ Construct a short-term rate tree under the **Ho-Lee Model** with time-dependent drift.
- ✓ Describe uses and benefits of the **arbitrage-free models** and assess the issue of **fitting models to market prices**.
- ✓ Describe the process of constructing a simple and recombining tree for a short-term rate under the **Vasicek Model with mean reversion**.
- ✓ Calculate the **Vasicek Model rate change**, **standard deviation** of the rate change, **expected rate** in T years, and *half-life*.
- ✓ Describe the **effectiveness** of the Vasicek Model.