2025 FRM®
Exam Prep

SchweserNotes™
Quantitative Analysis

Part I Book 2

KAPLAN SCHWESER

Book 2: Quantitative Analysis

SchweserNotes™ 2025

FRM Part I

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Readings and Learning Objectives

STUDY SESSION 4

12. Fundamentals of Probability

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 1.

After completing this reading, you should be able to:

a. describe an event and an event space.
b. describe independent events and mutually exclusive events.
c. explain the difference between independent events and conditionally independent events.
d. calculate the probability of an event for a discrete probability function.
e. define, describe, and calculate a conditional probability.
f. differentiate between conditional and unconditional probabilities.
g. explain and apply Bayes' rule.

13. Random Variables

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 2.

After completing this reading, you should be able to:

a. describe and differentiate a probability mass function from a cumulative distribution function and explain the relationship between these two.
b. describe and apply the concept of a mathematical expectation of a random variable.
c. describe the four common population moments.
d. explain the differences between a probability mass function and a probability density function.
e. describe the quantile function and quantile-based estimators.
f. explain the effect of a linear transformation of a random variable on the mean, variance, standard deviation, skewness, kurtosis, median, and interquartile range.

14. Common Univariate Random Variables

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 3.

After completing this reading, you should be able to:

a. illustrate the key properties and applications of the following distributions: Bernoulli distribution, binomial distribution, Poisson distribution, uniform distribution, normal distribution, lognormal distribution, Chi-squared distribution, Student's t distribution, F distribution, exponential distribution, and the Beta distribution.
b. construct mixture distributions, and explain the creation and characteristics of mixture distributions.

15. Multivariate Random Variables

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 4.

After completing this reading, you should be able to:

a. explain how a probability matrix can be used to express a probability mass function.
b. calculate the marginal and conditional distributions of a discrete bivariate random variable.
c. explain how the expectation of a function is calculated for a bivariate discrete random variable.
d. define covariance and explain what it measures.
e. explain the relationship between the covariance and correlation of two random variables, and how these are related to the independence of the two variables.
f. explain and illustrate the effects of applying linear transformations on the covariance and correlation between two random variables.
g. calculate the variance of a weighted sum of two random variables.
h. calculate the conditional expectation of a component of a bivariate random variable.
i. describe the features of an independent and identically distributed (iid) sequence of random variables.

j. explain how the iid property is helpful in calculating the mean and variance of a sum of iid random variables.

STUDY SESSION 5

16. Sample Moments

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 5.

After completing this reading, you should be able to:

a. estimate the mean, variance, and standard deviation using sample data.
b. explain the difference between a population moment and a sample moment.
c. differentiate between an estimator and an estimate.
d. describe the bias of an estimator and explain what the bias measures.
e. explain what is meant by the statement that the mean estimator is BLUE.
f. describe the consistency of an estimator and explain the usefulness of this concept.
g. explain how the Law of Large Numbers (LLN) and Central Limit Theorem (CLT) apply to the sample mean.
h. estimate and interpret the skewness and kurtosis of a random variable.
i. estimate quantiles, including the median, using sample data.
j. estimate the mean of two variables and apply the CLT.
k. estimate the covariance and correlation between two random variables.
l. explain how coskewness and cokurtosis are related to skewness and kurtosis.

17. Hypothesis Testing

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 6.

After completing this reading, you should be able to:

a. construct an appropriate null hypothesis and alternative hypothesis and differentiate between the two.
b. differentiate between a one-sided and a two-sided test and identify when to use each test.
c. explain the difference between Type I and Type II errors and how these relate to the size and power of a test.
d. explain how a hypothesis test and a confidence interval are related.
e. explain what the *p*-value of a hypothesis test measures.
f. construct and apply confidence intervals for one-sided and two-sided hypothesis tests and interpret the results of hypothesis tests with a specific confidence level.
g. identify the steps to test a hypothesis about the difference between two population means.
h. explain the problem of multiple testing and how it can lead to biased results.

STUDY SESSION 6

18. Linear Regression

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 7.

After completing this reading, you should be able to:

a. describe the models which can be estimated using linear regression and differentiate them from those which cannot.
b. interpret the results of an ordinary least squares (OLS) regression with a single explanatory variable.
c. describe the key assumptions of OLS parameter estimation.
d. describe the properties of OLS estimators and their sampling distributions.
e. construct, apply, and interpret hypothesis tests and confidence intervals for a single regression coefficient in a regression.
f. explain the steps needed to perform a hypothesis test in a linear regression.
g. describe the relationship among a *t*-statistic, its *p*-value, and a confidence interval.
h. estimate the correlation coefficient from the R^2 measure obtained in linear regressions with a single explanatory variable.

19. Regression with Multiple Explanatory Variables

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 8.

After completing this reading, you should be able to:

a. differentiate between the relative assumptions of single and multiple regression.
b. interpret regression coefficients in a multiple regression.
c. interpret goodness-of-fit measures for single and multiple regressions, including R^2 and adjusted-R^2.
d. construct, apply, and interpret joint hypothesis tests and confidence intervals for multiple coefficients in a regression.
e. calculate the regression R^2 using the three components of the decomposed variation of the dependent variable data: the explained sum of squares, the total sum of squares, and the residual sum of squares.

20. Regression Diagnostics

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 9.

After completing this reading, you should be able to:

a. explain how to test whether a regression is affected by heteroskedasticity.
b. describe approaches to using heteroskedastic data.
c. explain the concept of multicollinearity and differentiate between multicollinearity and perfect collinearity.
d. describe and illustrate the consequences of excluding a relevant explanatory variable from a model, and contrast those with the consequences of including an irrelevant regressor.
e. explain two model selection procedures and how these relate to the bias-variance trade-off.
f. describe the various methods of visualizing residuals and their relative strengths.
g. describe methods for identifying outliers and their impact.
h. determine the conditions under which OLS is the best linear unbiased estimator.

STUDY SESSION 7

21. Stationary Time Series

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 10.

After completing this reading, you should be able to:

a. describe the requirements for a series to be covariance stationary.
b. define the autocovariance function and the autocorrelation function.
c. define white noise, and describe independent white noise and normal (Gaussian) white noise.
d. define and describe the properties of autoregressive (AR) processes.
e. define and describe the properties of moving average (MA) processes.
f. explain how a lag operator works.
g. explain mean reversion and calculate a mean-reverting level.
h. define and describe the properties of autoregressive moving average (ARMA) processes.
i. describe the application of AR, MA, and ARMA processes.
j. describe sample autocorrelation and partial autocorrelation.
k. describe the Box-Pierce Q statistic and the Ljung-Box Q statistic.
l. explain how forecasts are generated from ARMA models.
m. describe the role of mean reversion in long-horizon forecasts.
n. explain how seasonality is modeled in a covariance-stationary ARMA.

22. Non-Stationary Time Series

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 11.

After completing this reading, you should be able to:

a. describe linear and nonlinear time trends.
b. explain how regression analysis can be used to model seasonality.
c. describe a random walk and a unit root.
d. explain the challenges of modeling time series containing unit roots.

e. describe how to test if a time series contains a unit root.
f. explain how to construct an h-step-ahead point forecast for a time series with seasonality.
g. calculate the estimated trend value and construct an interval forecast for a time series.

23. Measuring Returns, Volatility, and Correlation

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 12.

After completing this reading, you should be able to:

a. calculate, differentiate, and convert between simple and continuously compounded returns.
b. define and differentiate between volatility, variance rate, and implied volatility.
c. describe how the first two moments may be insufficient to describe non-normal distributions.
d. calculate the Jarque-Bera test statistic and explain how it is used to determine whether returns are normally distributed.
e. describe the power law and its use for non-normal distributions.
f. define correlation and covariance and differentiate between correlation and dependence.
g. describe properties of correlations between normally distributed variables when using a one-factor model.
h. compare and contrast the different measures of correlation used to assess dependence.

24. Simulation and Bootstrapping

Global Association of Risk Professionals. *Quantitative Analysis.* New York, NY: Pearson, 2022. Chapter 13.

After completing this reading, you should be able to:

a. describe the basic steps to conduct a Monte Carlo simulation and illustrate how this simulation method is used to approximate moments or other quantities.
b. describe ways to reduce Monte Carlo sampling error.
c. explain the use of antithetic and control variates in reducing Monte Carlo sampling error.
d. describe the bootstrapping method and its advantage over Monte Carlo simulation.
e. describe pseudo-random number generation.
f. describe situations where the bootstrapping method is ineffective.
g. describe the disadvantages of the simulation approach to financial problem solving.

25. Machine Learning Methods

Global Association of Risk Professionals. *Quantitative Analysis*. New York, NY: Pearson, 2022. Chapter 14.

After completing this reading, you should be able to:

a. discuss the philosophical and practical differences between machine learning techniques and classical econometrics.
b. compare and apply the two methods utilized for rescaling variables in data preparation.
c. explain the differences among the training, validation, and test data sub-samples, and how each is used.
d. examine the differences between and consequences of underfitting and overfitting, and describe potential remedies for each.
e. explain how principal components analysis is used to reduce the dimensionality of a set of features.
f. describe how the K-means algorithm separates a sample into clusters.
g. explain the mechanics behind natural language processing and how it is used.
h. differentiate among unsupervised, supervised, and reinforcement learning models.
i. explain how reinforcement learning operates, and calculate Q-values utilized in the decision-making process.

26. Machine Learning and Prediction

Global Association of Risk Professionals. *Quantitative Analysis*. New York, NY: Pearson, 2022. Chapter 15.

After completing this reading, you should be able to:

a. explain the role of linear regression and logistic regression in prediction.
b. evaluate the predictive performance of logistic regression models.
c. describe and apply methods used to encode categorical variables.
d. discuss why regularization is useful, and compare the ridge regression and LASSO approaches.
e. illustrate how a decision tree is constructed and interpreted.
f. describe how ensembles of learners are built.

g. explain the intuition and processes behind the K-nearest neighbors and support vector machine methods for classification.
h. explain how neural networks are constructed and how their weights are determined.
i. compare the logistic regression and neural network classification approaches using a confusion matrix.

The following is a review of the Quantitative Analysis principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Quantitative Analysis, Chapter 1.

READING 12

FUNDAMENTALS OF PROBABILITY

Study Session 4

EXAM FOCUS

This reading covers important terms and concepts associated with probability theory. Specifically, we will examine the difference between independent and mutually exclusive events, discrete probability functions, and the difference between unconditional and conditional probabilities. Bayes' rule is also examined as a way to update a given set of prior probabilities. For the exam, be able to calculate conditional probabilities, joint probabilities, and probabilities based on a probability function. Also, understand when and how to apply Bayes' formula.

MODULE 12.1: BASICS OF PROBABILITY

When an outcome is unknown, such as the outcome (realization) of the flip of a coin or the high temperature tomorrow in Dubai, we refer to it as a **random variable**. We can describe a random variable with the probabilities of its possible outcomes. For the flip of a fair coin, we refer to the probability of heads as P(heads), which is 50%. We can think of a probability as the likelihood that an outcome will occur. If we flip a fair coin 100 times, we expect that on average it will be heads 50 times.

A probability equal to 0 for an outcome means that the outcome will not happen. A probability equal to 1 for an outcome means it will happen with certainty. Probabilities cannot be less than 0 or greater than 1.

The probability that a random variable will have a specific outcome, given that some other outcome has occurred, is referred to as a **conditional probability**. The probability that A will occur, given that B has occurred, is written as P(A|B). For example, the probability that a day's high temperature in Seattle will be between 70 and 80 degrees is an **unconditional probability** (i.e., *marginal probability*). The probability that the high temperature will be between 70 and 80 degrees, given that the sky is cloudy that day, is a conditional probability.

The probability that both A and B will occur is written P(AB) and referred to as the **joint probability** of A and B (both occurring).

Events and Event Spaces

LO 12.a: Describe an event and an event space.

An **event** is a single outcome or a combination of outcomes for a random variable. Consider a random variable that is the result of rolling a fair six-sided die. The outcomes with positive probability (those that may happen) are the integers 1, 2, 3, 4, 5, and 6. For the event x = 3, we can write P(3) = 1/6 = 16.7%. Other possible events include getting a 3 *or* 4, P(3 or 4) = 2/6 = 33.3%, and getting an even number, P(x is even) = P(x = 2, 4, or 6) = 3/6 = 50%. The probability that the realization of this random variable is equal to one of the possible outcomes (x = 1, 2, 3, 4, 5, or 6) is 100%.

The **event space** for a random variable is the set of all possible outcomes and combinations of outcomes. Consider a flip of a fair coin. The event space is heads, tails, heads and tails, and neither heads nor tails. P(heads) and P(tails) are both 50%. The probability of both heads and tails is zero, as is the probability of neither heads nor tails.

PROFESSOR'S NOTE

The notation $P(A \cup B)$ is sometimes used to mean the probability of A *or* B, and the notation $P(A \cap B)$ is sometimes used to mean the probability of A *and* B.

Independent and Mutually Exclusive Events

LO 12.b: Describe independent events and mutually exclusive events.

Two events are **independent events** if knowing the outcome of one does not affect the probability of the other. When two events are independent, the following two probability relationships must hold:

1. P(A) × P(B) = P(AB). The probability that both A and B will happen is the product of their unconditional probabilities.
2. P(A|B) = P(A). The conditional probability of A given that B occurs is simply the unconditional probability of A occurring. This means B occurring does not change the probability of A.

Consider flipping a coin twice. Getting heads on the first flip does not change the probability of getting heads on the second flip. The two events are independent. In this case, the **joint probability** of getting heads on both flips is simply the product of their unconditional expectations. Given that the probability of getting heads is 50%, the probability of getting heads on two flips in a row is 0.5 × 0.5 = 25%.

If A_1 , A_2 , ..., A_n are independent events, their joint probability P(A_1 and A_2 ... and A_n) is equal to P(A_1) × P(A_2) × ... × P(A_n).

Two events are **mutually exclusive events** if they cannot both happen. Consider the possible outcomes of one roll of a die. The events "x = an even number" and "x = 3" are mutually exclusive; they cannot both happen on the same roll.

In general, $P(A \text{ or } B) = P(A) + P(B) - P(AB)$. We must subtract the probability of both A and B happening to avoid counting those outcomes twice. If the probability that one stock will rise tomorrow, P(A), is 60% and the probability that another stock will rise tomorrow, P(B), is 55%, we cannot calculate the probability that both will rise tomorrow as 60% + 55% = 115%. We must subtract the joint probability that both stocks will rise to get P(A or B).

When events A and B are mutually exclusive, P(AB) is zero, so P(A or B) is simply $P(A) + P(B)$.

Conditionally Independent Events

LO 12.c: Explain the difference between independent events and conditionally independent events.

Two conditional probabilities, $P(A|C)$ and $P(B|C)$, may be independent or dependent regardless of whether the unconditional probabilities, P(A) and P(B), are independent or not. When two events are **conditionally independent events**, $P(A|C) \times P(B|C) = P(AB|C)$.

Consider Event A, "scores above average on an exam," and Event B, "is taller than average." For a population of grade school students, these events may not be independent, as taller students are older on average and likely in a higher grade. Taller students may well do better on a given exam than shorter (younger) students. If we add the conditioning Event C "age equals 8," we may find that height and exam scores are independent, that is, $P(A|C)$ and $P(B|C)$ are independent while P(A) and P(B) are not.

MODULE QUIZ 12.1

1. For the roll of a fair six-sided die, how many of the following are classified as events?
   - The outcome is 3.
   - The outcome is an even number.
   - The outcome is not 2, 3, 4, 5, or 6.

   A. One.
   B. Two.
   C. Three.
   D. None.

2. Which of the following equalities does not imply that the events A and B are independent?
   A. $P(AB) = P(A) \times P(B)$.
   B. $P(A \text{ or } B) = P(A) + P(B) - P(AB)$.
   C. $P(A|B) = P(A)$.
   D. $P(AB) / P(B) = P(A)$.

3. Two independent events:
   A. must be conditionally independent.
   B. cannot be conditionally independent.
   C. may be conditionally independent or not conditionally independent.

D. are conditionally independent only if they are mutually exclusive events.

MODULE 12.2: CONDITIONAL, UNCONDITIONAL, AND JOINT PROBABILITIES

Discrete Probability Function

LO 12.d: Calculate the probability of an event for a discrete probability function.

A **discrete probability function** is one for which there are a finite number of possible outcomes. The probability function gives us the probability of each possible outcome. Consider a random variable for which the possible outcomes are x = 1, 2, 3, or 4, with a probability function of x/10 so that P(x) = x/10. The probability of an outcome of 3 is 3/10 = 30%. The probability of an outcome of either 2 or 4 is 2/10 + 4/10 = 60%. This function qualifies as a probability function because the probability of getting one of the possible outcomes is 1/10 + 2/10 + 3/10 + 4/10 = 10/10 = 100%.

Conditional and Unconditional Probabilities

LO 12.e: Define, describe, and calculate a conditional probability.

LO 12.f: Differentiate between conditional and unconditional probabilities.

Sometimes we are interested in the probability of an event, given that some other event has occurred. As mentioned earlier, we refer to this as a **conditional probability**, P(A|B).

Consider conditional probabilities that an employee at Acme, Inc., earns more than $40,000 per year, P(40+), conditioned on the highest level of education an employee has attained. Employees fall into one of three education levels: no degree (ND), bachelor's degree (BD), and higher-than-bachelor's degree (HBD). If 60% of the employees have no degree, 30% of the employees have attained only a bachelor's degree, and 10% have attained a higher degree, we write P(ND) = 60%, P(BD) = 30%, and P(HBD) = 10%.

Note that the three levels of education attainment are *mutually exclusive*; an employee can only be in one of the three categories of educational attainment. Note also that the three categories are also *exhaustive*; the categories cover all the possible levels of educational attainment. We can write this as P(ND or BD or HBD) = 100%.

Given a conditional probability and the unconditional probability of the conditioning event, we can calculate the **joint probability** of both events using P(AB) = P(A|B) × P(B). Assume that for Acme, 10% of the employees with no degree, 70% of the employees with only a bachelor's degree, and 100% of employees with a degree beyond a bachelor's degree earn more than $40,000 per year. That is, P(40+|ND) = 10%, P(40+|BD) = 70%, and P(40+|HBD) = 100%.

Using these conditional probabilities, along with the unconditional probabilities P(ND) = 60%, P(BD) = 30%, and P(HBD) = 10%, we can calculate the joint probabilities:

$$P(40+ \text{ and ND}) = 10\% \times 60\% = 6\%$$
$$P(40+ \text{ and BD}) = 70\% \times 30\% = 21\%$$
$$P(40+ \text{ and HBD}) = 100\% \times 10\% = 10\%.$$

We can use these probabilities to illustrate the **total probability rule**, which states that if the conditioning events B_i are mutually exclusive and exhaustive then:

$$P(A) = P(A | B_1)P(B_1) + P(A | B_2)P(B_2) + \dots + P(A | B_n) P(B_n)$$

This is the sum of the joint probabilities. For Acme, we have P(40+) = 6% + 21% + 10% = 37% of the employees earn more than $40,000 per year.

Rearranging P(AB) = P(A|B) × P(B), we get:

$$P(A | B) = \frac{P(AB)}{P(B)}$$

That is, we can calculate a conditional probability from the joint probability of two events and the unconditional probability of the conditioning event. As an example, the conditional probability is P(40+|BD) is:

$$\frac{P(40 + \text{ and BD})}{P(BD)} = \frac{21\%}{30\%} = 70\%$$

Bayes' Rule

LO 12.g: Explain and apply Bayes' rule.

Bayes' rule allows us to use information about the outcome of one event to improve our estimates of the unconditional probability of another event.

From our rules of probability, we know that P(A|B) × P(B) = P(AB) and that P(B|A) × P(A) = P(AB), so we can write P(A|B) × P(B) = P(B|A) × P(A). Rearranging these terms, we can arrive at Bayes' rule:

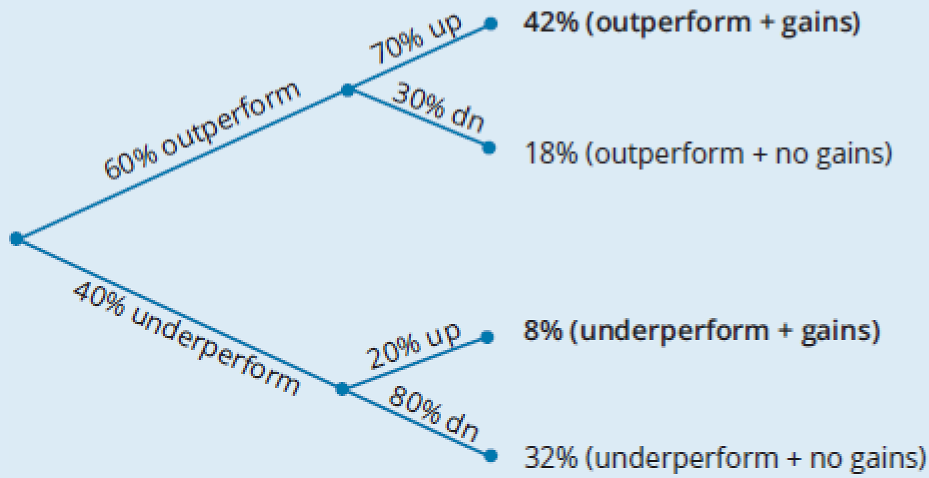
$$P(A | B) = \frac{P(B | A) \times P(A)}{P(B)}$$

Given the unconditional probabilities of A and B and the conditional probability of B given A, we can calculate the conditional probability of A given B. The following example illustrates the use of Bayes' rule and provides some intuition about what this formula is telling us.

EXAMPLE: Bayes' formula

There is a 60% probability the economy will outperform, and if it does, there is a 70% probability a stock will go up and a 30% probability the stock will go down. There is a 40% probability the economy will underperform, and if it does, there is a 20% probability the stock in question will increase in value (have gains) and an 80% probability it will not. Given that the stock increased in value, **calculate** the probability that the economy outperformed.

Answer:



In the earlier figure, we have multiplied the probabilities to calculate the probabilities of each of the four outcome pairs. Note that these sum to 1. Given that the stock has gains, what is our updated probability of an outperforming economy? We sum the probability of stock gains in both states (outperform and underperform) to get 42% + 8% = 50%. Given that the stock has gains, the probability that the economy has outperformed is:

$$\frac{42\%}{50\%} = 84\%$$

The numerator for the calculation of the updated probability P(A|B) using Bayes' formula in the example is the joint probability of outperform and gains. This is calculated as P(gains|outperform) × P(outperform) (i.e., 0.7 × 0.6 = 0.42). The denominator is the unconditional probability of gains, P(gains|outperform) + P(gains|underperform) (i.e., 0.42 + 0.08 = 0.50).

EXAMPLE: Probability concepts and relationships

A shipment of 1,000 cars has been unloaded into a parking area. The cars have the following features:

- There are 600 blue (B) cars.
- Of the blue cars, 150 have driver assist (DA) technology.
- There are 400 red (R) cars.
- Of the red cars, 200 have DA technology.

Given these facts, **calculate** the following:

1. Unconditional probabilities: P(B) and P(R)
2. Conditional probabilities: P(DA|B) and P(DA|R)
3. Joint probabilities: P(B and DA) and P(R and DA)
4. Total probability rule: P(DA)
5. Bayes' rule: P(B|DA)

Answer:

Unconditional probabilities:

P(B) = 600/1,000 = 60%

P(R) = 400/1,000 = 40%

Conditional probabilities:

P(DA|B) = 150/600 = 25%

P(DA|R) = 200/400 = 50%

Joint probabilities:

P(B and DA) = P(DA|B)P(B) = 25%(60%) = 15%; 15%(1,000) = 150 of the cars are blue with driver assist

P(R and DA) = P(DA|R)P(R) = 50%(40%) = 20%; 20%(1,000) = 200 of the cars are red with driver assist

Total probability rule:

P(DA) = P(DA|B)P(B) + P(DA|R)P(R) = 25%(60%) + 50%(40%) = 35%; 35%(1,000) = 350 of the cars have driver assist

Bayes' rule:

P(B|DA) = P(B and DA)/P(DA) = 15%/35% = 42.9%; 350 cars have driver assist and of those cars, 150 are blue: 150/350 = 0.42857 = 42.9%

Independence:

Now, assume we add to our information that 40% of the blue cars (240) are convertibles and 40% of the red cars (160) are convertibles, so that 400 of the cars are convertibles. In this case, P(B|C) = 240/400 = 60% = P(B) and P(R|C) = 160/400 = 40% = P(R). This meets the requirement for independence that P(A|B) = P(A). The fact that a car chosen at random is a convertible gives us no additional information about whether a car is blue or red.

MODULE QUIZ 12.2

1. The probability function for the outcome of one roll of a six-sided die is given as P(X) = x/21. What is P(x > 4)?
   A. 16.6%.
   B. 23.8%.
   C. 33.3%.
   D. 52.4%.

2. The relationship between the probability that both Event A and Event B will occur and the conditional probability of Event A given that Event B occurs is:
   A. P(AB) = P(A|B)P(B).
   B. $P(A) = \frac{P(A|B)}{P(AB)}$.

P(AB)

C. $P(A) = \frac{P(AB)}{P(A|B)}$.

D. P(AB) = P(A|B)P(A).

3. The probability that shares of Acme will increase in value over the next month is 50% and the probability that shares of Acme and shares of Best will both increase in value over the next month is 40%. The probability that Best shares will increase in value, given that Acme shares increase in value over the next month, is closest to:
   A. 20%.
   B. 40%.
   C. 80%.
   D. 90%.

KEY CONCEPTS

LO 12.a

An event is one of the possible outcomes or a subset of the possible outcomes of a random event, such as the flip of a coin. The event space is all the subsets of possible outcomes and the empty set (none of the possible outcomes).

LO 12.b

Two events are independent if either of the following conditions hold:

- P(A) × P(B) = P(AB)
- P(A|B) = P(A)

Two events are mutually exclusive if the joint probability, P(AB) = 0 (i.e., both cannot occur). When two events are mutually exclusive, P(A or B) = P(A) + P(B).

LO 12.c

If two events conditional on a third event are independent, we say they are conditionally independent. For example, if P(AB|C) = P(A|C) P(B|C), then A and B are conditionally independent. Two events may be independent but conditionally dependent, or vice versa.

LO 12.d

A probability function describes the probability for each possible outcome for a discrete probability distribution. For example, P(x) = x/25, defined over the outcomes {1, 2, 3, 4, 5}.

LO 12.e

The joint probability of two events, P(AB), is the probability that they will both occur: P(AB) = P(A|B) × P(B). This relationship can be rearranged to define the conditional probability of A given B as follows:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

LO 12.f

An unconditional probability (i.e., marginal probability) is the probability of an event occurring.

A conditional probability, P(A|B), is the probability of an Event A occurring given that Event B has occurred.

LO 12.g

Bayes' rule is:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

This formula allows us to update the unconditional probability, P(A), based on the fact that B has occurred. P(AB) can be calculated as P(B|A)P(A).

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 12.1

1. **C** All of the outcomes and combinations specified are included in the event space for the random variable. (LO 12.a)

2. **B** P(A or B) = P(A) + P(B) – P(AB) holds for both independent and dependent events. The other equalities are only true for independent events. (LO 12.b)

3. **C** Two independent events may be conditionally independent or not conditionally independent. (LO 12.c)

Module Quiz 12.2

1. **D** The probability of $x > 4$ is the probability of an outcome of 5 or 6 (5/21 + 6/21 = 52.4%). (LO 12.d)

2. **A** The (joint) probability that both A and B will occur is equal to the conditional probability of Event A given that Event B has occurred, multiplied by the unconditional probability of Event B. (LO 12.e)

3. **C** Bayes' formula tells us that:

   $$P(A \mid B) = \frac{P(AB)}{P(B)}$$

   Applying that to the information given, we can write:

   $$P(\text{Best increases} \mid \text{Acme increases}) = \frac{P(\text{Best increases and Acme increases})}{P(\text{Acme increases})}$$

   40%/50% = 80%

   (LO 12.g)

The following is a review of the Quantitative Analysis principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Quantitative Analysis, Chapter 2.

READING 13

RANDOM VARIABLES

Study Session 4

EXAM FOCUS

This reading addresses the concepts of expected value, variance, skewness, and kurtosis. The characteristics and calculations of these measures will be discussed. For the exam, be able to distinguish among a probability mass function, a cumulative distribution function, and a probability density function. Also, be able to compute expected value, and be able to identify the four common population moments of a statistical distribution.

MODULE 13.1: PROBABILITY MASS FUNCTIONS, CUMULATIVE DISTRIBUTION FUNCTIONS, AND EXPECTED VALUES

Random Variables and Probability Functions

LO 13.a: Describe and differentiate a probability mass function from a cumulative distribution function and explain the relationship between these two.

A **discrete random variable** is one that can take on only a countable number of possible outcomes. An example of a discrete random variable is a Bernoulli random variable that takes on only two possible values, zero and one. We can model the outcome of a coin flip as a Bernoulli random variable where heads = 1 and tails = 0. The number of days in June that will have a temperature greater than 70 degrees is also a discrete random variable. The possible outcomes are the integers from 0 to 30.

A **continuous random variable** has an uncountable number of possible outcomes. The amount of rainfall that will fall in June is an example of a continuous random variable. There are an infinite number of possible outcomes because for any two values (e.g., 6.95 inches and 6.94 inches), we can find a number between them [e.g., (6.95 + 6.94) / 2 = 6.945]. Because there are an infinite number of possible outcomes, the probability of any single value is zero. For continuous random variables, we measure probabilities only over some positive interval, (e.g., the probability that rainfall in June will be between 6.94 and 6.95 inches).

A **probability mass function (PMF)**, $f(x) = P(X = x)$, gives us the probability that the outcome of a discrete random variable, X, will be equal to a given number, x. For a Bernoulli random variable for which the $P(x = 1) = p$, the PMF is $f(x) = p^x (1 - p)^{1-x}$. This yields $P(x = 1) = p$ and $P(x = 0) = 1 - p$.

A second example of a PMF is f(x) = 1/6, which is the probability that one roll of a six-sided die will take on one of the possible outcomes one through six. Each of the possible outcomes has the same probability of occurring (1/6 = 16.67%).

A third example is the PMF $f(x) = x/10$ for a random variable that can take on values of 1, 2, 3, or 4. For example, $P(x = 3) = f(3) = 3/10 = 30\%$.

For all of these PMFs, the sum of the probabilities of all of the possible outcomes is 100%, a requirement for a PMF.

A **cumulative distribution function (CDF)** gives us the probability that a random variable will take on a value less than or equal to *x* [i.e., $F(x) = P(X \leq x)$].

For a Bernoulli random variable with possible outcomes of zero and one, the CDF is:

$$F(x) = \begin{matrix} 0 & x < 0 \\ 1 - p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{matrix}$$

While the PMF for this Bernoulli variable is defined only for X = 0 or 1, the corresponding CDF is defined for all real numbers. For example, $P(X < 0.1456) = F(0.1456) = 1 - p$.

For the roll of a six-sided die, the CDF is F(x) = x/6, so that the probability of a roll of 3 or less is F(3) = 3/6 = 50. This illustrates an important relationship between a PMF and its corresponding CDF; the probability of an outcome less than or equal to *x* is simply the sum of the probabilities of all the possible outcomes less than or equal to *x*. For the roll of a six-sided die. $F(3) = f(1) + f(2) + f(3) = 1/6 + 1/6 + 1/6 = 3/6 = 50\%$.

Expectations

LO 13.b: Describe and apply the concept of a mathematical expectation of a random variable.

The **expected value** is the weighted average of the possible outcomes of a random variable, where the weights are the probabilities that the outcomes will occur. The mathematical representation for the expected value of random variable *X* is:

$$E(X) = \Sigma P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$$

Here, *E* is referred to as the expectations operator and is used to indicate the computation of a probability-weighted average. The symbol x_1 represents the first observed value (observation) for random variable *X*; x_2 is the second observation, and so on through the *n*th observation. The concept of expected value may be demonstrated using probabilities associated with a coin toss. On the flip of one coin, the occurrence of the event "heads" may be used to assign the value of one to a random variable.

Alternatively, the event "tails" means the random variable equals zero. Statistically, we would formally write the following:

if heads, then $X = 1$

if tails, then $X = 0$

For a fair coin, P(heads) = P(X = 1) = 0.5, and P(tails) = P(X = 0) = 0.5. The expected value can be computed as follows:

$$E(X) = \Sigma P(x_i)x_i = P(X = 0)(0) + P(X = 1)(1) = (0.5)(0) + (0.5)(1) = 0.5$$

In any individual flip of a coin, *X* cannot assume a value of 0.5. Over the long term, however, the average of all the outcomes is expected to be 0.5. Similarly, the expected value of the roll of a fair die, where X = number that faces up on the die, is determined to be:

$$E(X) = \Sigma P(x_i)x_i = (1/6)(1) + (1/6)(2) + (1/6)(3) + (1/6)(4) + (1/6)(5) + (1/6)(6)$$

$$E(X) = 3.5$$

We can never roll a 3.5 on a die, but over the long term, 3.5 should be the average value of all outcomes.

The expected value is, statistically speaking, our best guess of the outcome of a random variable. While a 3.5 will never appear when a die is rolled, the average amount by which our guess differs from the actual outcomes is minimized when we use the expected value calculated this way.

Note that the probabilities of the outcomes for a coin flip (0 or 1) and the probabilities of the outcomes for the roll of a die are equal for all of the possible outcomes in both cases. When outcomes are equally likely, the expected value is simply the mean (average) of the outcomes:

$$\frac{1+0}{2} = 0.5 \text{ for a coin flip}$$

$$\frac{1+2+3+4+5+6}{6} = 3.5 \text{ for the roll of a die}$$

When we estimate the expected value of a random variable based on *n* observations, we use the mean of the observed values as our estimate of the mean of the underlying probability distribution. In terms of a probability model, we are assuming that the outcomes are equally likely, that is, each has a probability of 1/n. Multiplying each outcome by 1/n and then summing them, produces the same expected value as dividing the sum of the outcomes by *n*.

In other cases, the probabilities of the outcomes are not equal and we calculate the expected value as the weighted sum of the outcomes, where the weights are the probabilities of each outcome. The following example illustrates such a case.

EXAMPLE: Expected earnings per share (EPS)

The probability distribution of EPS for Ron's Stores is given in the following figure. **Calculate** the expected earnings per share.

EPS Probability Distribution

Probability	EPS
10%	£1.80
20%	£1.60
40%	£1.20
30%	£1.00
100%	

Answer:

The expected EPS is simply a weighted average of each possible EPS, where the weights are the probabilities of each possible outcome.

$$E(EPS) = 0.10(1.80) + 0.20(1.60) + 0.40(1.20) + 0.30(1.00) = £1.28$$

The following are two useful properties of expected values:

1. If *c* is any constant, then:

   $E(cX) = cE(X)$

2. If *X* and *Y* are any random variables, then:

   $E(X + Y) = E(X) + E(Y)$

MODULE QUIZ 13.1

1. The probability mass function (PMF) for a discrete random variable that can take on the values 1, 2, 3, 4, or 5 is $P(X = x) = x/15$. The value of the cumulative distribution function (CDF) of 4, F(4), is equal to:
   A. 26.7%.
   B. 40.0%.
   C. 66.7%.
   D. 75.0%.

2. An analyst has estimated the following probabilities for gross domestic product growth next year:

   P(4%) = 10%, P(3%) = 30%, P(2%) = 40%, P(1%) = 20%

   Based on these estimates, the expected value of GDP growth next year is:
   A. 2.0%.
   B. 2.3%.
   C. 2.5%.
   D. 2.8%.

MODULE 13.2: MEAN, VARIANCE, SKEWNESS, AND KURTOSIS

LO 13.c: Describe the four common population moments.

The population moments most often used are

- mean,
- variance,